Section 2.2: Inverse of a Matrix

Consider the scalar equation \( ax = b \). Provided \( a \neq 0 \), we can solve this explicitely

\[ x = a^{-1} b \]

where \( a^{-1} \) is the unique number such that \( aa^{-1} = a^{-1} a = 1 \).

If \( A \) is an \( n \times n \) matrix, we seek an analog \( A^{-1} \) that satisfies the condition

\[ A^{-1} A = AA^{-1} = I_n. \]

If such matrix \( A^{-1} \) exists, we’ll say that \( A \) is nonsingular (a.k.a. invertible). Otherwise, we’ll say that \( A \) is singular.
Theorem (2 \times 2 case)

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( ad - bc \neq 0 \), then \( A \) is invertible and

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

If \( ad - bc = 0 \), then \( A \) is singular.

The quantity \( ad - bc \) is called the \textbf{determinant} of \( A \) and may be denoted in several ways

\[
\text{det}(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.
\]
Find the inverse if possible

(a) \[ A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \]

(b) \[ A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \]
Theorem

If $A$ is an invertible $n \times n$ matrix, then for each $b$ in $\mathbb{R}^n$, the equation $Ax = b$ has unique solution $x = A^{-1}b$. 
Example

Solve the system

\[\begin{align*}
3x_1 + 2x_2 &= -1 \\
-x_1 + 5x_2 &= 4
\end{align*}\]
Theorem

(i) If $A$ is invertible, then $A^{-1}$ is also invertible and

\[ (A^{-1})^{-1} = A. \]

(ii) If $A$ and $B$ are invertible $n \times n$ matrices, then the product $AB$ is also invertible\(^1\) with

\[ (AB)^{-1} = B^{-1}A^{-1}. \]

(iii) If $A$ is invertible, then so is $A^T$. Moreover

\[ (A^T)^{-1} = (A^{-1})^T. \]

\(^1\)This can generalize to the product of $k$ invertible matrices.
Elementary Matrices

Definition: An elementary matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products $E_1A$, $E_2A$, and $E_3A$.

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$
\[
A = \begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{bmatrix}
\]

\[
E_3 = \begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\]
Remarks

- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),

- Each elementary matrix is invertible where the inverse *undoes* the row operation,

- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

\[
\text{rref}(A) = E_k \cdots E_2 E_1 A.
\]
Theorem

An $n \times n$ matrix $A$ is invertible if and only if it is row equivalent to the identity matrix $I_n$. Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n,$$

then

$$A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1}\right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces $A$ to $I_n$, transforms $I_n$ into $A^{-1}$.

This last observation—operations that take $A$ to $I_n$ also take $I_n$ to $A^{-1}$—gives us a method for computing an inverse!
Algorithm for finding $A^{-1}$

To find the inverse of a given matrix $A$:

- Form the $n \times 2n$ augmented matrix $[A \ I]$.
- Perform whatever row operations are needed to get the first $n$ columns (the $A$ part) to rref.
- If rref($A$) is $I$, then $[A \ I]$ is row equivalent to $[I \ A^{-1}]$, and the inverse $A^{-1}$ will be the last $n$ columns of the reduced matrix.
- If rref($A$) is NOT $I$, then $A$ is not invertible.

Remarks: We don’t need to know ahead of time if $A$ is invertible to use this algorithm.
If $A$ is singular, we can stop as soon as it’s clear that rref($A$) $\neq I$. 
Examples: Find the Inverse if Possible

(a) \[
\begin{bmatrix}
1 & 2 & -1 \\
-4 & -7 & 3 \\
-2 & -6 & 4 \\
\end{bmatrix}
\]
Examples: Find the Inverse if Possible

(b) \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 4 \\
5 & 6 & 0 \\
\end{bmatrix}
\]
Solve the linear system if possible

\[
\begin{align*}
  x_1 + 2x_2 + 3x_3 &= 3 \\
  x_2 + 4x_3 &= 3 \\
  5x_1 + 6x_2 &= 4
\end{align*}
\]
Section 2.3: Characterization of Invertible Matrices

Given an \( n \times n \) matrix \( A \), we can think of

- A matrix equation \( Ax = b; \)
- A linear system that has \( A \) as its coefficient matrix;
- A linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by \( T(x) = Ax; \)
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

**Question:** How is this stuff related, and how does being singular or invertible tie in?
Theorem: Suppose $A$ is $n \times n$. The following are equivalent.\(^2\)

(a) $A$ is invertible.
(b) $A$ is row equivalent to $I_n$.
(c) $A$ has $n$ pivot positions.
(d) $Ax = 0$ has only the trivial solution.
(e) The columns of $A$ are linearly independent.
(f) The transformation $x \mapsto Ax$ is one to one.
(g) $Ax = b$ is consistent for every $b$ in $\mathbb{R}^n$.
(h) The columns of $A$ span $\mathbb{R}^n$.
(i) The transformation $x \mapsto Ax$ is onto.
(j) There exists an $n \times n$ matrix $C$ such that $CA = I$.
(k) There exists an $n \times n$ matrix $D$ such that $AD = I$.
(l) $A^T$ is invertible.

\(^2\)Meaning all are true or none are true.
Theorem: (An inverse matrix is unique.)

Let $A$ and $B$ be $n \times n$ matrices. If $AB = I$, then $A$ and $B$ are both invertible with $A^{-1} = B$ and $B^{-1} = A$. 
Invertible Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that both

$$S(T(x)) = x \text{ and } T(S(x)) = x$$

for every $x$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$
Theorem (Invertibility of a linear transformation and its matrix)

Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear transformation and \( A \) its standard matrix. Then \( T \) is invertible if and only if \( A \) is invertible. Moreover, if \( T \) is invertible, then

\[
T^{-1}(x) = A^{-1}x
\]

for every \( x \) in \( \mathbb{R}^n \).
Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

\[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \text{ given by } T(x_1, x_2) = (3x_1 - x_2, 4x_2). \]
Example

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether $T$ is onto? Why (or why not)?