February 21 MATH 1112 sec. 52 Spring 2020

Trigonometric Functions of Any Angle

Reference Angles: Suppose we want to find the trig values for the angle θ shown. Note that the acute angle (pink) has terminal side through (x, y), and by symmetry the terminal side of θ passes through the point $(-x, y)$ (same y and opposite sign x).

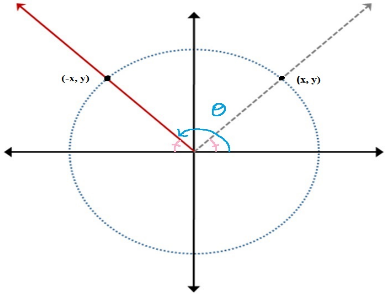
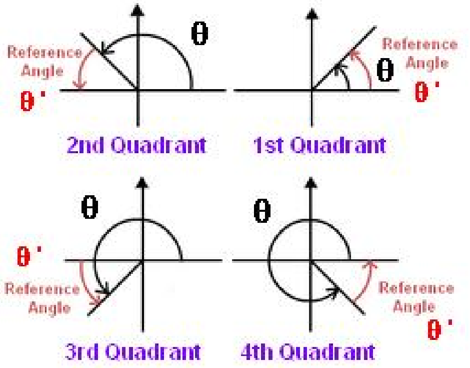


Figure: What is the connection between the trig values for θ and those for the acute angle in pink?

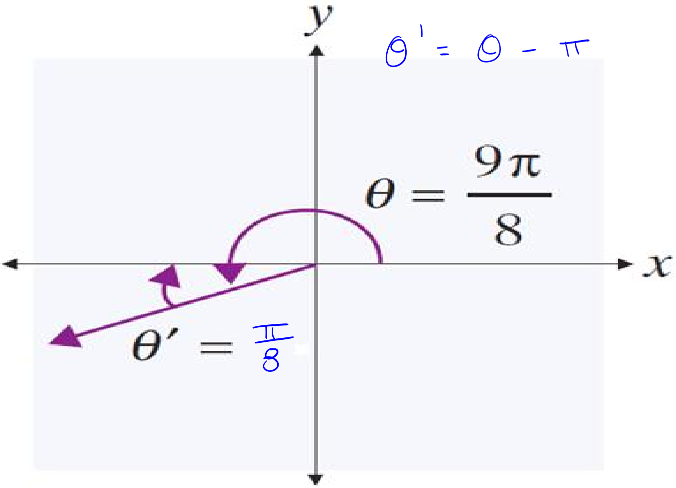
Reference Angles

Definition: Let θ be an angle in standard position that is not a quadrantal angle. The **reference angle** θ' associated with θ is the angle of measure $0^\circ < \theta' < 90^\circ$ between the terminal side of θ and the *nearest* part of the x-axis.



Example (a)

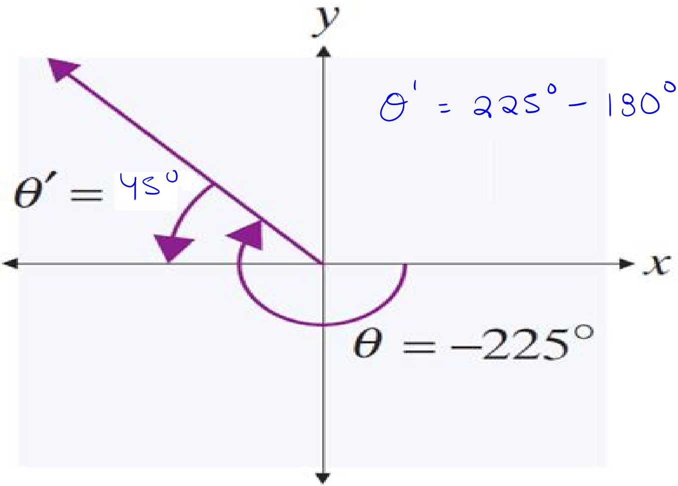
Determine the reference angle.



$\theta' = \theta - \pi$

Example (b)

Determine the reference angle.



Theorem on Reference Angles

Theorem: If θ' is the reference angle for the angle θ, then

$$\sin\theta' = |\sin\theta|, \quad \cos\theta' = |\cos\theta| \quad \& \quad \tan\theta' = |\tan\theta|.$$

Remark 1: The analogous relationships hold for the cosecant, secant, and cotangent.

Remark 2: This means that the trigonometric values for θ can differ at most by a sign (+ or -) from the values for θ'.

Recall The Trigonometric Values

θ°	0°	30°	45°	60°	90°
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

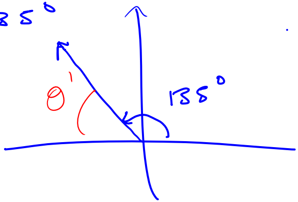
Remember the sign mnemonic:

All **S**tudents **T**ake **C**alculus

Example: Using Reference Angles

Find the exact value of

Draw 135°



(a) $\sin(135^\circ)$

$\sin(135^\circ) = \pm \sin(45^\circ)$

$\theta' = 180^\circ - 135^\circ$
$= 45^\circ$

135° is a quad II angle so sine is positive.

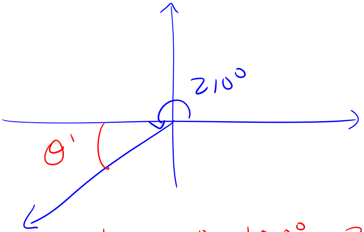
$\sin 45^\circ = \frac{1}{\sqrt{2}}$

$\sin(135^\circ) = \frac{1}{\sqrt{2}}$

Example: Using Reference Angles

Find the exact value of

Draw 210°



(b) $\cos(210^\circ)$

$\cos(210^\circ) = \pm\cos(30^\circ)$

210° is a quad III angle

Cosine is negative

$\cos(210^\circ) = \frac{-\sqrt{3}}{2}$

$\theta' = 210^\circ - 180^\circ = 30^\circ$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$

More New Trigonometric Identities

Quotient Identities: For any given θ for which both sides are defined

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \& \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

If (x,y) is a point on the terminal side of θ in standard position, and $r = \sqrt{x^2+y^2} > 0$

$\sin\theta = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$

$$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{r}\cdot\frac{r}{x} = \frac{y}{x} = \tan\theta$$

Example

Use the given information to determine the remaining trigonometric values of θ.

$$\sin\theta = \frac{1}{4} \quad \text{and} \quad \cos\theta = -\frac{\sqrt{15}}{4}$$

$\csc\theta = \dfrac{1}{\sin\theta} = 4$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{-4}{\sqrt{15}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{1}{4}}{-\frac{\sqrt{15}}{4}} = \dfrac{1}{4}\left(\dfrac{-4}{\sqrt{15}}\right) = \dfrac{-1}{\sqrt{15}}$

$\cot\theta = \dfrac{1}{\tan\theta} = -\sqrt{15}$