February 21 Math 1190 sec. 62 Spring 2017

Chapter 2: Derivatives

Let's recall the definitions and derivative rules we have so far:

Let's assume that $y=f(x)$ is a function with c in it's domain. The **derivative** of f at c is

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

provided the limit exists.

Notes on Notation

If $y = f(x)$, we have the Leibniz notation for the derivative function

$$f'(x) = \frac{dy}{dx}.$$

We can use this notation for the derivative **at a point** in the following well accepted way

$$f'(c) = \left.\frac{dy}{dx}\right|_c .$$

For example, if $y = x^2$, then

$$\frac{dy}{dx} = 2x, \quad \text{and} \quad \left.\frac{dy}{dx}\right|_3 = 6.$$

The Derivative Function

For $y = f(x)$, the derivative

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Another, sometimes convenient formulation of this is

$$\frac{dy}{dx} = f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$$

using a **dummy variable** (a place keeper) z.

Derivative Rules

Constant: $\frac{d}{dx}c = 0$

Identity: $\frac{d}{dx}x = 1$

Power: $\frac{d}{dx}x^n = nx^{n-1}$ for integers n

Exponential: $\frac{d}{dx}e^x = e^x$

Derivative Rules

Constant Factor: $\frac{d}{dx}kf(x) = kf'(x)$

Sum or Difference: $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Product: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Quotient: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Six Trig Function Derivatives

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$

$$\frac{d}{dx}\sec x = \sec x\tan x, \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

Question

Suppose $y = 2xe^x$. Evaluate $\left.\frac{dy}{dx}\right|_0$

(a) $\left.\frac{dy}{dx}\right|_0 = 0$

(b) $\left.\frac{dy}{dx}\right|_0 = 2$

(c) $\left.\frac{dy}{dx}\right|_0 = 2e$

(d) $\left.\frac{dy}{dx}\right|_0 = e$

$$\frac{dy}{dx} = 2\cdot 1\, e^x + 2\cdot x\, e^x$$

$$\left.\frac{dy}{dx}\right|_0 = 2\cdot 1\, e^0 + 2\cdot 0\cdot e^0 = 2$$

Question

True or False: If f is differentiable at c, then the slope, m_{tan}, of the line tangent to the graph of f at the point $(c, f(c))$ is

$$m_{tan} = f'(c).$$

Derivative, Slope of tangent line, rate of change

all the same concept.

Question

Using the quotient rule, $\dfrac{d}{dx}\dfrac{\cos x}{x^3+2} =$

(a) $\dfrac{-\sin x}{3x^2}$

(b) $\dfrac{-(x^3+2)\sin x - 3x^2\cos x}{x^6+4}$

(c) $\dfrac{(x^3+2)\sin x + 3x^2\cos x}{(x^3+2)^2}$

(d) $\dfrac{-(x^3+2)\sin x - 3x^2\cos x}{(x^3+2)^2}$

Section 3.1: The Chain Rule

Suppose we wish to find the derivative of $f(x) = (x^2 + 2)^2$.

We can expand f first

$$f(x) = x^4 + 4x^2 + 4$$

$$f'(x) = 4x^3 + 8x$$

Compositions

Now suppose we want to differentiate $G(x) = (x^2 + 2)^{10}$. How about $F(x) = \sqrt{x^2 + 2}$?

We could expand G, work out that 10^{th} power. F can't be "exponded;" the only way to take F' is using the definition.

Thus far

Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \sqrt{x^2+2} = (f \circ g)(x).$$

The composition matches the "order of operations" from the inside out.

Take $g(x) = x^2+2$, $f(u) = \sqrt{u}$

$$(f \circ g)(x) = f(g(x)) = f(x^2+2) = \sqrt{x^2+2} = F(x)$$

as expected!

Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \cos\left(\frac{\pi x}{2}\right) = (f \circ g)(x).$$

$g(x) = \frac{\pi}{2}x$ and $f(u) = \cos u$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{\pi}{2}x\right) = \cos\left(\frac{\pi}{2}x\right)$ ✓

Theorem: Chain Rule

Suppose g is differentiable at x and f is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at x and

$$\frac{d}{dx}F(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example

Determine any inside and outside functions and find the derivative.

(a) $F(x) = \sin^2 x = (\sin x)^2$

$u = g(x) = \sin x$, $g'(x) = \cos x$

$f(u) = u^2$, $f'(u) = 2u$

$*$

$(f \circ g)(x) = f(\sin x)$

$= (\sin x)^2$

$= \sin^2 x$

$F'(x) = f'(g(x)) \cdot g'(x)$

$= 2(\sin x) \cos x = 2 \sin x \cos x$

(b) $F(x) = e^{x^4-5x^2+1}$

$u = g(x) = x^4 - 5x^2 + 1$, $g'(x) = 4x^3 - 10x$

$f(u) = e^u$, $f'(u) = e^u$

$* \ f(g(x)) = f(x^4 - 5x^2 + 1)$

$= e^{x^4-5x^2+1}$

$F'(x) = f'(g(x)) \cdot g'(x)$

$= e^{x^4-5x^2+1} \cdot (4x^3 - 10x)$

$= (4x^3 - 10x)\, e^{x^4-5x^2+1}$

Question

If $G(\theta) = \cos\left(\frac{\pi\theta}{2} - \frac{\pi}{4}\right) = f(g(\theta))$ where

$$f(u) = \cos u, \quad \text{and} \quad g(\theta) = \frac{\pi\theta}{2} - \frac{\pi}{4}, \quad \text{then}$$

$g'(\theta) = \frac{\pi}{2}$

$f'(u) = -\sin u$

(a) $G'(\theta) = -\frac{\pi}{2}\sin\theta$

(b) $G'(\theta) = -\frac{\pi}{2}\sin\left(\frac{\pi\theta}{2} - \frac{\pi}{4}\right)$

(c) $G'(\theta) = -\sin\left(\frac{\pi\theta}{2}\right) + \sin\left(\frac{\pi}{4}\right)$

The power rule with the chain rule

If $u = g(x)$ is a differentiable function and n is any integer, then

$$\frac{d}{dx}\,u^n = nu^{n-1}\frac{du}{dx}.$$

Evaluate: $\frac{d}{dx}e^{7x} = \frac{d}{dx}\,(e^x)^7$

$= 7(e^x)^6 \cdot e^x$

$= 7e^{6x} \cdot e^x$

$= 7e^{7x}$

$u = e^x$ and $n = 7$

$\frac{du}{dx} = e^x$

* This can be verified by taking $e^{7x} = f(g(x))$ where $f(u) = e^u$ and $g(x) = 7x$ *

Question

Evaluate $\frac{d}{dx}\sin^4(x) = \frac{d}{dx}(\sin x)^4$ $= 4\,(\sin x)^3 \cdot \cos x$

$n\,u^{n-1} \cdot \frac{du}{dx}$

(a) $4\cos^3(x)$

(b) $4\sin^3(x)\cos^3(x)$

(c) $4\sin^3(x)\cos(x)$

(d) $-4\sin^3(x)\cos(x)$

Example

Find the equation of the line tangent to the graph of $y = \cos^2 x$ at the point $(\frac{\pi}{4}, \frac{1}{2})$.

We need m_{tan}. $\quad m_{tan} = \left.\frac{dy}{dx}\right|_{\frac{\pi}{4}}$

$$y = (\cos x)^2, \quad u = \cos x \quad f(u) = u^2$$

$$\frac{du}{dx} = -\sin x \quad f'(u) = 2u$$

$$\Rightarrow \frac{dy}{dx} = 2\cos x(-\sin x) = -2\cos x \sin x$$

$$m_{tan} = -2\cos\frac{\pi}{4}\sin\frac{\pi}{4} = -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-2}{2} = -1$$

$$y - \frac{1}{2} = -1\left(x - \frac{\pi}{4}\right) \Rightarrow \boxed{y = -x + \frac{\pi}{4} + \frac{1}{2}}$$

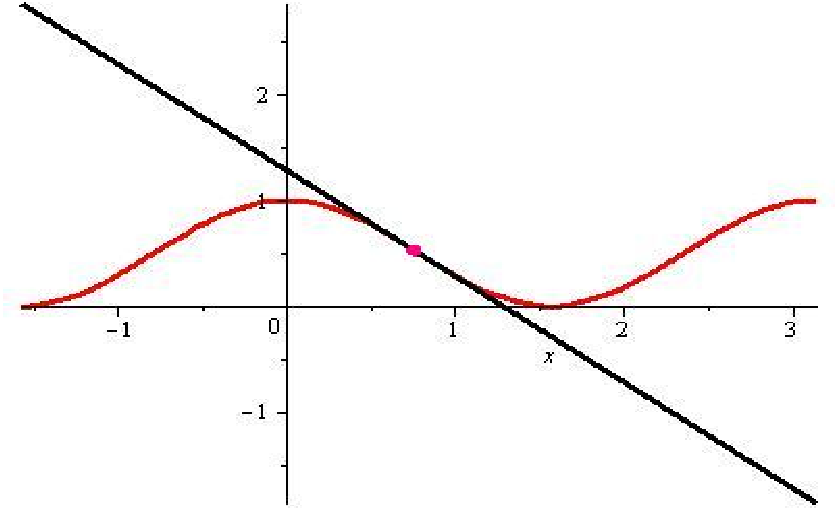

Figure: $y = \cos^2 x$ and the tangent line at $(\frac{\pi}{4}, \frac{1}{2})$.

We may be able to choose between differentiation methods.

Evaluate $\dfrac{d}{dx}\,\dfrac{\sin x}{x^3+2}$ using

(a) The quotient rule:

$$\frac{d}{dx}\frac{\sin x}{x^3+2} = \frac{\cos x\,(x^3+2) - \sin x\,(3x^2)}{(x^3+2)^2} = \frac{(x^3+2)\cos x - 3x^2\sin x}{(x^3+2)^2}$$

$$= \frac{(x^3+2)\cos x}{(x^3+2)^2} - \frac{3x^2\sin x}{(x^3+2)^2}$$

$$= \frac{\cos x}{(x^3+2)} - \frac{3x^2\sin x}{(x^3+2)^2}$$

(b) writing $\dfrac{\sin x}{x^3+2} = (\sin x)(x^3+2)^{-1}$ and using the chain rule.

Power rule

$$\frac{d}{dx}\frac{\sin x}{x^3+2} = (\cos x)(x^3+2)^{-1} + (\sin x)\left[-1(x^3+2)^{-2}\cdot(3x^2+0)\right]$$

$$= \frac{\cos x}{(x^3+2)} - (\sin x)(3x^2)(x^3+2)^{-2}$$

$$= \frac{\cos x}{x^3+2} - \frac{3x^2\sin x}{(x^3+2)^2}$$

same as before

Question

Suppose that $g(x)$ is differentiable. The derivative of $e^{g(x)}$ is (Hint: The *outside* function is e^u, and the *inside* function is $g(x)$.)

(a) $e^{g(x)}\, g'(x)$

(b) $e^{g(x)}$

(c) $e^{g'(x)}$

$y = f(u) \quad u = g(x) \quad \text{where} \quad f(u) = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= e^u \cdot g'(x) = e^{g(x)} g'(x)$$

Question

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x)$$

Find the equation of the line tangent to the graph of $f(x) = e^{\sin x}$ at the point $(0, f(0))$.

(a) $y = x + 1$

Details left as an exercise

(b) $y = 1$

(c) $y = x - 1$

(d) $y = ex + 1$

Multiple Compositions

The chain rule can be iterated to account for multiple compositions. For example, suppose f, g, h are appropriately differentiable, then

$$\frac{d}{dx}(f\circ g\circ h)(x)=\frac{d}{dx}f(g(h(x))=f'(g(h(x))\,g'(h(x))\,h'(x)$$

Note that the outermost function is f, and its inner function is a composition $g(h(x))$.

So the derivative of the outer function evaluated at the inner is $f'(g(h(x))$ which is multiplied by the derivative of the inner function—**itself based on the chain rule**—$g'(h(x))h'(x)$.

Example

Evaluate the derivative $\dfrac{d}{dt}\tan^2\left(\dfrac{1}{3}t^3\right) = \dfrac{d}{dt}\left(\tan\left(\frac{1}{3}t^3\right)\right)^2$

$v = h(t) = \frac{1}{3}t^3 \quad , \quad h'(t) = t^2$

$u = g(v) = \tan(v) \quad , \quad g'(v) = \sec^2 v$

$f(u) = u^2 \quad , \quad f'(u) = 2u$

$\frac{d}{dt}\tan^2\left(\frac{1}{3}t^3\right) = 2u \cdot \sec^2 v \cdot t^2 = 2\tan v \sec^2 v \cdot t^2$

$= 2\tan\left(\frac{1}{3}t^3\right)\sec^2\left(\frac{1}{3}t^3\right)\cdot t^2$

Exponential of Base a

Let $a > 0$ with $a \neq 1$. By properties of logs and exponentials

$$a^x = e^{(\ln a)x}. = e^{\ln a^x}$$

$(\ln a)x$ is a constant, $\ln a$, times x. $\frac{d}{dx}(\ln a)x = \ln a$

So $\frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = e^{(\ln a)x} \cdot \ln a$

$= a^x \ln a$

Theorem: (Derivative of $y = a^x$) Let $a > 0$ and $a \neq 1$. Then

$$\frac{d}{dx} a^x = a^x \ln a$$

Exponential of Base a

Theorem: (Derivative of $y = a^x$) Let $a > 0$ and $a \neq 1$. Then

$$\frac{d}{dx}a^x = a^x \ln a$$

Recall from before that if $f(x) = a^x$ then

$$\frac{d}{dx}a^x = f'(0)a^x \quad \text{where} \quad f'(0) = \lim_{h\to 0}\frac{a^h - 1}{h}.$$

We now see that the number $f'(0) = \ln(a)$.

Example

Evaluate

(a) $\dfrac{d}{dx}4^x$ $= 4^x \ln 4$ here $a = 4$

(b) $\dfrac{d}{dx}2^{\cos x}$ $= 2^{\cos x} \ln 2 \, (-\sin x)$

$= -(\sin x)\, 2^{\cos x} \ln 2$

inner $g(x) = \cos x = u$

$g'(x) = -\sin x$

$f(u) = 2^u$, $f'(u) = 2^u \ln 2$

Question

$$\frac{d}{dx}7^{x^2} =$$

(a) 7^{x^2}

(b) $(2x)7^{x^2}$

(c) $(2x)7^{x^2}\ln 7$

(d) $7^{x^2}\ln 7$