February 22 Math 2306 sec. 57 Spring 2018

Section 8: Homogeneous Equations with Constant Coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad a,b,c\text{—constant}$$

has characteristic equation is $am^2+bm+c=0$.

If the roots of the characteristic equation are

- $m_1\neq m_2$ (distinct and real), the general solution

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

- $m_1=m_2=m$ (one repeated real), the general solution

$$y=c_1e^{mx}+c_2xe^{mx}$$

- $m=\alpha\pm i\beta$ (complex conjugates), the general solution

$$y=c_1e^{\alpha x}\cos(\beta x)+c_2e^{\alpha x}\sin(\beta x)$$

Solve the initial value problem

$$y'' + 6y' + 13y = 0 \quad y(0) = 0, \quad y'(0) = 1$$

Char. Eqn $\quad m^2 + 6m + 13 = 0$

we'll complete the square

$$m^2 + 6m + 9 - 9 + 13 = 0$$

$$(m^2 + 6m + 9) + 4 = 0$$

$$(m+3)^2 + 4 = 0$$

$$(m+3)^2 = -4 \implies m+3 = \pm\sqrt{-4} = \pm 2i$$

$$m = -3 \pm 2i \qquad \alpha = -3, \quad \beta = 2$$

$$y_1 = e^{-3x}\cos(2x), \quad y_2 = e^{-3x}\sin(2x)$$

General solution

$$y = c_1 e^{-3x} \cos(2x) + c_2 e^{-3x} \sin(2x)$$

Impose $y(0) = 0$, $y'(0) = 1$

$$y' = -3c_1 e^{-3x} \cos(2x) - 2c_1 e^{-3x} \sin(2x) - 3c_2 e^{-3x} \sin(2x) + 2c_2 e^{-3x} \cos(2x)$$

$$y(0) = c_1 e^{0} \cos(0) + c_2 e^{0} \sin(0) = 0$$

$$c_1 + 0 = 0 \Rightarrow c_1 = 0$$

$$y'(0) = -3\underset{0}{\cancel{c_1}} e^{0} \cos(0) - 2\underset{0}{\cancel{c_1}} e^{0} \sin(0) - 3c_2 e^{0} \sin(0) + 2c_2 e^{0} \cos(0) = 1$$

$$2c_2 = 1$$

$$c_2 = \frac{1}{2}$$

The solution to the IVP is

$$y = \frac{1}{2} e^{-3x} \sin(2x)$$

Find the general solution of the ODE

$$\frac{d^2u}{dt^2} - \frac{du}{dt} - u = 0$$

Characteristic eqn

$$m^2 - m - 1 = 0$$

quadratic formula

$$m = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

2 real roots $m_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $m_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$

$$y_1 = e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)x}, \quad y_2 = e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)x}$$

The general solution

$$y = c_1 e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)x}$$

Higer Order Linear Constant Coefficient ODEs

- If a root m is real repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

- If $m = \alpha \pm i\beta$ is a repeated pair of complex conjugate roots repeated k times, we get $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Find the general solution of the ODE

$y''' - 3y'' + 3y' - y = 0$

Characteristic eqn

$$m^3 - 3m^2 + 3m - 1 = 0$$

This is $(m-1)^3 = 0$

$m = 1$, triple root

$y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2e^x$

General solution

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$