

Section 8: Homogeneous Equations with Constant Coefficients

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0, \quad a, b, c \text{---constant}$$

has characteristic equation is $am^2 + bm + c = 0$.

If the roots of the characteristic equation are

- $m_1 \neq m_2$ (distinct and real), the general solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

- $m_1 = m_2 = m$ (one repeated real), the general solution

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

- $m = \alpha \pm i\beta$ (complex conjugates), the general solution

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

Solve the initial value problem

$$y'' + 6y' + 13y = 0 \quad y(0) = 0, \quad y'(0) = 1$$

Find the general solution to the ODE.

Characteristic eqn

$$m^2 + 6m + 13 = 0 .$$

Complete the square

$$m^2 + 6m + 9 - 9 + 13 = 0$$

$$(m^2 + 6m + 9) + 4 = 0$$

$$(m+3)^2 + 4 = 0$$

$$(m+3)^2 = -4$$

$$m+3 = \pm\sqrt{-4} = \pm 2i$$

$$m = -3 \pm 2i \quad \alpha = -3, \beta = 2$$

$$y_1 = e^{-3x} \cos(2x), \quad y_2 = e^{-3x} \sin(2x)$$

Gen. solution

$$y = C_1 e^{-3x} \cos(2x) + C_2 e^{-3x} \sin(2x)$$

Now impose the conditions $y(0) = 0, y'(0) = 1$

$$y'(x) = -3C_1 e^{-3x} \cos(2x) - 2C_1 e^{-3x} \sin(2x) - 3C_2 e^{-3x} \sin(2x) + 2C_2 e^{-3x} \cos(2x)$$

$$y(0) = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) = 0$$

$$C_1 + 0 = 0 \Rightarrow C_1 = 0$$

$$y'(0) = -3C_1 e^0 \cos(0) - 2C_1 e^0 \sin(0) - 3C_2 e^0 \sin(0) + 2C_2 e^0 \cos(0) = 1$$

$\downarrow \quad \quad \quad \downarrow$

$$0 + 2C_2 = 1$$

$$C_2 = \frac{1}{2}$$

The solution to the IVP

$$y = \frac{1}{2} e^{-3x} \sin(2x)$$

Find the general solution of the ODE

$$\frac{d^2u}{dt^2} - \frac{du}{dt} - u = 0$$

Characteristic eqn

$$m^2 - m - 1 = 0$$

quadratic formula

$$m = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

2 real roots

$$m_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad m_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

$$u_1 = e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)t}, \quad u_2 = e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)t}$$

The general solution

$$u = C_1 e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)t} + C_2 e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)t}$$

Higer Order Linear Constant Coefficient ODEs

- If a root m is real repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \dots, \quad x^{k-1} e^{mx}$$

- If $m = \alpha \pm i\beta$ is a repeated pair of complex conjugate roots repeated k times, we get $2k$ solutions

$$e^{\alpha x} \cos(\beta x), \quad e^{\alpha x} \sin(\beta x), \quad xe^{\alpha x} \cos(\beta x), \quad xe^{\alpha x} \sin(\beta x), \quad \dots,$$

$$x^{k-1} e^{\alpha x} \cos(\beta x), \quad x^{k-1} e^{\alpha x} \sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Find the general solution of the ODE

$$y''' - 3y'' + 3y' - y = 0$$

Characteristic eqn

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m = 1 \quad \text{triple root}$$

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x$$

General solution

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$