February 22 Math 3260 sec. 56 Spring 2018

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has A as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: How is this stuff related, and how does being singular or invertible tie in?

Theorem: Suppose A is $n \times n$. The following are equivalent. [1]

(a) A is invertible.
(b) A is row equivalent to I_n.
(c) A has n pivot positions.
(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The columns of A are linearly independent.
(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.
(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
(j) There exists an $n \times n$ matrix C such that $CA = I$.
(k) There exists an $n \times n$ matrix D such that $AD = I$.
(l) A^T is invertible.

[1] Meaning all are true or none are true.

Theorem: (An inverse matrix is unique.)

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

To show this, we consider the homogeneous equation $B\vec{x} = \vec{0}$. Multiply both sides on the left by A.

$$A B \vec{x} = A\vec{0} \implies I\vec{x} = \vec{0}$$

$$\implies \vec{x} = \vec{0}.$$

The homogeneous equation has only the trivial solution. By the previous theorem, B is invertible — i.e. B^{-1} exists.

From $AB = I$, multiply on the right by B^{-1}.

$$ABB^{-1} = IB^{-1} \Rightarrow AI = B^{-1}$$

$$\Rightarrow A = B^{-1}.$$

Since B^{-1} is invertible, we know that A is invertible and

$$A^{-1} = \left(B^{-1}\right)^{-1} = B.$$

Invertible Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Theorem (Invertibility of a linear transformation and its matrix)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \text{given by} \quad T(x_1, x_2) = (3x_1 - x_2, 4x_2).$$

Construct standard matrix A.

$T(\vec{e}_1) = T(1,0) = (3,0)$, $T(\vec{e}_2) = T(0,1) = (-1,4)$

$$A = [T(\vec{e}_1) \ T(\vec{e}_2)] = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$$

$\det(A) = 3 \cdot 4 - 0(-1) = 12$ $\quad$ $\det(A) \neq 0$, A^{-1} exists

Since A^{-1} exists, T is invertible and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$$

$$T^{-1}(\vec{x}) = A^{-1}\vec{x} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 4x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}x_1 + \frac{1}{12}x_2 \\ \frac{1}{4}x_2 \end{bmatrix}$$

$$T^{-1}(x_1, x_2) = \left(\frac{1}{3}x_1 + \frac{1}{12}x_2, \frac{1}{4}x_2\right)$$

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?

The standard matrix A, for T, is $n \times n$. We're given $\vec{x} \mapsto A\vec{x}$ is one to one. This is statement (f) in our theorem. By statement (i) $\vec{x} \mapsto A\vec{x}$ is onto. That is, T is also onto.

Section 3.1: Introduction to Determinants

Recall that a 2×2 matrix is invertible if and only if the number called its **determinant** is nonzero. We had

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

We wish to extend the concept of determinant to $n \times n$ matrices in general. And we wish to do so in such a way that invertibility holds if and only if the determinant is nonzero.

Determinant 3×3 case:

Suppose we start with a 3×3 invertible matrix. And suppose that $a_{11} \neq 0$. We can multiply the second and third rows by a_{11} and begin row reduction.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Determinant 3×3 case continued...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

If $A \sim I$, one of the entries in the $2,2$ or the $3,2$ position must be nonzero. Let's assume it is the $2,2$ entry. Continue row reduction to get

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}.$$

Again, if A is invertible, it must be that the bottom right entry is nonzero. That is

$$\Delta \neq 0.$$

Note that if $\Delta = 0$, the rref of A is not I—A would be singular.

Determinant 3×3 case continued...

With a little rearrangement, we have

$$\begin{aligned}\Delta &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ &+ \quad a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\end{aligned}$$

The number Δ will be called the **determinant** of A.

Definitions: Minors

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

Definition: The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

Definitions: Cofactor

Definition: Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Example: Find the three minors M_{11}, M_{12}, M_{13} and find the 3 cofactors C_{11}, C_{12}, C_{13} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$M_{11} = \det(A_{11})$

$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

$M_{11} = a_{22}a_{33} - a_{32}a_{23}$

(Example Continued...)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$M_{12} = \det(A_{12})$

$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$

$M_{12} = a_{21}a_{33} - a_{31}a_{23}$

$M_{13} = \det(A_{13})$ $\quad A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$M_{13} = a_{21}a_{32} - a_{31}a_{22}$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = M_{11}$$

$$= a_{22} a_{33} - a_{32} a_{23}$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -M_{12}$$

$$= -(a_{21} a_{33} - a_{31} a_{23})$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = M_{13}$$

$$= a_{21} a_{32} - a_{31} a_{22}$$

Observation:

Comparison with the determinant of the 3×3 matrix, we can note that

$$\Delta = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$