February 23 Math 1190 sec. 62 Spring 2017

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

The chain rule states that for a differentiable composition $f(g(x))$

$$\frac{d}{dx}\, f(g(x)) = f'(g(x))g'(x)$$

For $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Example

Assume f is a differentiable function of x. Find an expression for the derivative:

$$\frac{d}{dx}\,(f(x))^2 = 2\left(f(x)\right)f'(x)$$

$$= 2f(x)\,f'(x)$$

Chain rule

$u = f(x)$, $\frac{du}{dx} = f'(x)$

$\frac{d}{du}u^2 = 2u$

$$\frac{d}{dx}\,\tan\left(f(x)\right) = \sec^2\left(f(x)\right)\cdot f'(x)$$

Here

$\frac{d}{du}\tan u = \sec^2 u$

Example

Suppose we know that $y = f(x)$ for some differentiable function (but we don't know exactly what f is). Find an expression for the derivative.

$$\frac{d}{dx}\, y^2 = 2y\,\frac{dy}{dx}$$

we do the same, except express $f'(x)$ as $\frac{dy}{dx}$

we need the product rule for the product x^2 times y^2

$$\frac{d}{dx}\, x^2y^2$$

$$= \underbrace{2x}_{f'}\,\underbrace{y^2}_{g} + \underbrace{x^2}_{f}\,\underbrace{(2y)\frac{dy}{dx}}_{g'}$$

$$= 2xy^2 + 2x^2y\,\frac{dy}{dx}$$

Question

If y is some function of x (but we don't know the specifics), then

$$\frac{d}{dx}y^3 =$$

(a) $\left(\frac{dy}{dx}\right)^3$

(b) $3y^2\frac{dy}{dx}$

(c) $3\left(\frac{dy}{dx}\right)^2$

Let's Double Check with an Example:

Let $y = x^2$ so that $\dfrac{dy}{dx} = 2x$ and $y^3 = x^6$.

Compute with the power rule $\dfrac{d}{dx}y^3 = \dfrac{d}{dx}x^6$ $= 6x^5$

Now take a moment and compute each of

(a) $\left(\dfrac{dy}{dx}\right)^3 = (2x)^3 = 8x^3$

(b) $3y^2\dfrac{dy}{dx} = 3(x^2)^2(2x) = 6x^4 \cdot x = 6x^5$

(c) $3\left(\dfrac{dy}{dx}\right)^2 = 3(2x)^2 = 3(4x^2) = 12x^2$

Match!

Implicitly defined functions

A relation—an equation involving two variables x and y—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

implies that y is defined to be one or more functions of x.

The first gives 2 functions:

$y^2 = 16 - x^2 \implies y = \sqrt{16 - x^2}$ or

$y = -\sqrt{16 - x^2}$

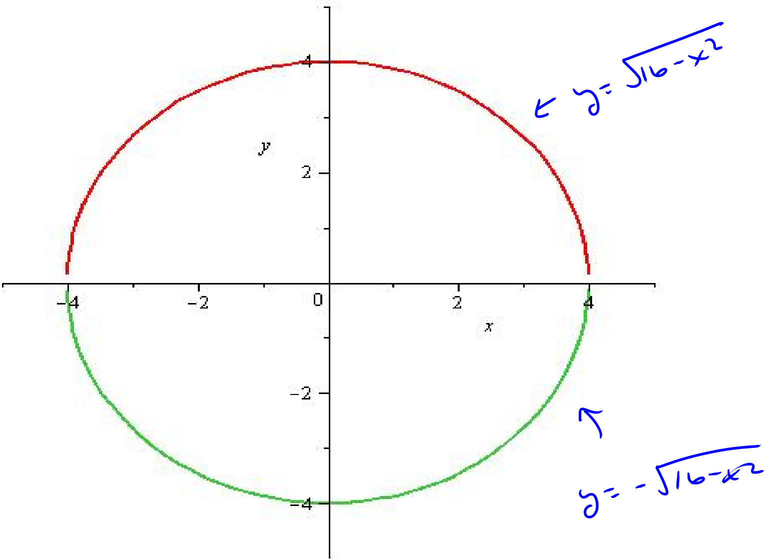

Figure: $x^2 + y^2 = 16$

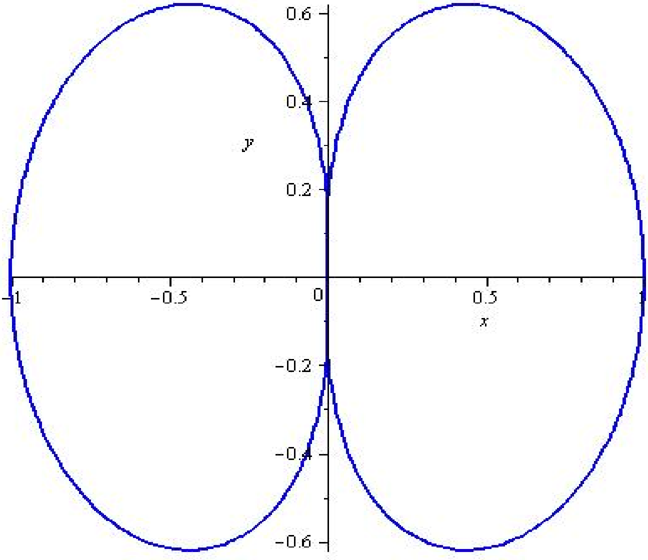

Figure: $(x^2 + y^2)^3 = x^2$

Explicit -vs- Implicit

A function is defined **explicitly** when given in the form

$$y = f(x).$$

e.g. $y = \tan x$ or $y = e^x$ or

$y = x^3 + xe^x$

A function is defined *implicitly* when it is given as a relation

$$F(x, y) = C,$$

for constant C.

e.g. $x^2 + y^2 = 16$ or $xe^y + ye^x = 0$

Implicit Differentiation

Since $x^2 + y^2 = 16$ *implies* that y is a function of x, we can consider it's derivative.

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 + y^2 = 16.$$

Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(16)$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}16 = 0$$

$$2x + 2y\frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} \implies \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Show that the same result is obtained knowing

$$y = \sqrt{16 - x^2} \quad \text{or} \quad y = -\sqrt{16 - x^2}.$$

I'll do the 2^{nd} one, the first is left as an exercise.

We need: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Let $y = -\sqrt{16 - x^2}$

$$\frac{dy}{dx} = -\left(\frac{1}{2\sqrt{16-x^2}}\right) \cdot (0 - 2x)$$

Chain rule

$$\frac{dy}{dx} = -\frac{-\cancel{2}x}{\cancel{2}\sqrt{16-x^2}} = -\frac{-x}{\sqrt{16-x^2}}$$

$$= -\frac{x}{-\sqrt{16-x^2}}$$

But $y = -\sqrt{16-x^2}$

$$= -\frac{x}{y}$$ as expected,

Example

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 - 3xy + y^2 = y.$$

product

$$\frac{d}{dx}\left(x^2 - 3xy + y^2\right) = \frac{d}{dx} y$$

$$2x - 3\left(1\cdot y + x\cdot\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$

$$2y\frac{dy}{dx} - 3x\frac{dy}{dx} - \frac{dy}{dx} = 3y - 2x$$

$$(2y - 3x - 1)\frac{dy}{dx} = 3y - 2x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3y - 2x}{2y - 3x - 1}}$$

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dS}{dr}$.

$\frac{dS}{dr}$ S-dependent, r-independent

$$e^{Sr} + S = r^2 + 2$$

$$\frac{d}{dr}\left(e^{Sr} + S\right) = \frac{d}{dr}\left(r^2+2\right)$$

$$e^{Sr}\frac{d}{dr}(Sr) + \frac{dS}{dr} = 2r$$

product

$$e^{Sr}\left(\frac{dS}{dr}r + S\cdot 1\right) + \frac{dS}{dr} = 2r$$

$$re^{Sr}\frac{dS}{dr} + Se^{Sr} + \frac{dS}{dr} = 2r$$

$$r e^{Sr} \frac{dS}{dr} + \frac{dS}{dr} = 2r - S e^{Sr}$$

$$\left(r e^{Sr} + 1\right) \frac{dS}{dr} = 2r - S e^{Sr}$$

$$\boxed{\frac{dS}{dr} = \frac{2r - S e^{Sr}}{r e^{Sr} + 1}}$$

Example

Find the equation of the line tangent to the graph of $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

$\frac{dy}{dx}$ still gives the slope of the tangent line. So we find $\frac{dy}{dx}$:

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(6xy\right)$$

product

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(1\cdot y + x\cdot\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

Divide by 3, collect $\frac{dy}{dx}$

$$y^2 \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - x^2$$

$$(y^2 - 2x)\frac{dy}{dx} = 2y - x^2$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

We need the slope @ (3,3). So we set $x=3$ and $y=3$

$$m_{tan} = \frac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = \frac{6-9}{9-6} = -1$$

point $(3,3)$ and slope -1

$$y-3=-1(x-3)$$

$$y=-x+3+3$$

$$\boxed{y=-x+6}$$

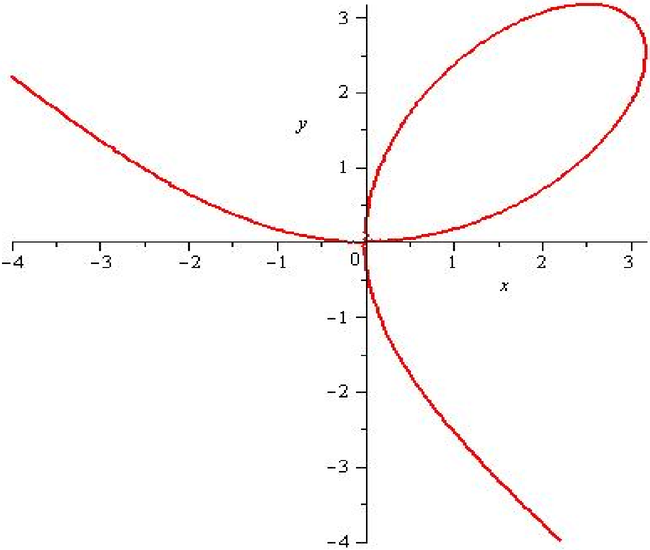

Figure: Folium of Descartes $x^3 + y^3 = 6xy$

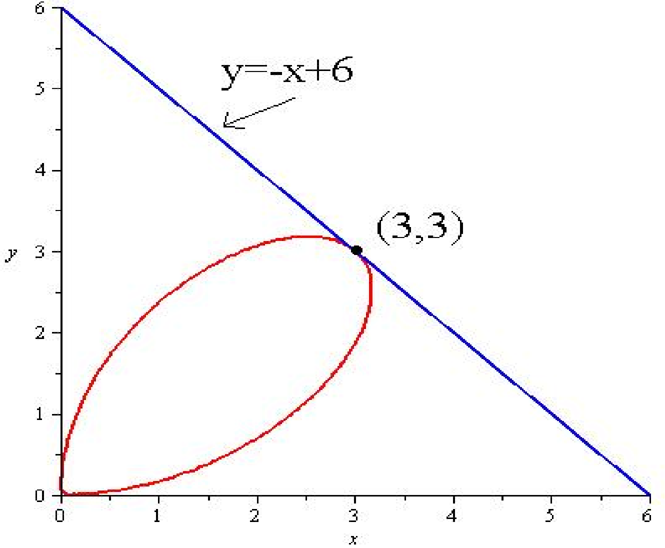

Figure: Folium of Descartes with tangent line at $(3, 3)$