Feb. 23 Math 2254H sec 015H Spring 2015

Section 10.1: Parametric Curves

Definition: We consider a **path** to be a curve along with an orientation—i.e. a specified direction of motion along that curve.

Definitions: t is called a **parameter**, the pair

$$x = f(t), \quad y = g(t)$$

is called a set of **parametric equations**, and the collection of points (x, y) is called a **parametric curve**.

Some Common Examples

Determine and plot the curve defined by

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

Try to write the curve in terms of x and y w/ out t. (Eliminate the parameter.)

$x^2 = \cos^2 t$ and $y^2 = \sin^2 t$

so $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

i.e. $x^2 + y^2 = 1$

Circle of radius 1 centered @ (0,0).

When $t=0$ $\quad x(0)=\cos(0)=1$ and $y(0)=\sin(0)=0$

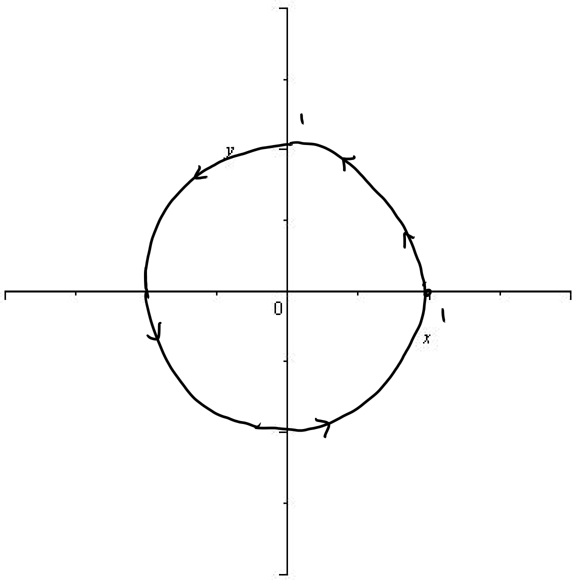
$t=2\pi$ $\quad x(2\pi)=1$ and $y(2\pi)=0$

When $t=\frac{\pi}{2}$

$x\left(\frac{\pi}{2}\right)=0$ $\quad y\left(\frac{\pi}{2}\right)=1$

when $t=\frac{\pi}{4}$

$x\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ $\quad y\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$


y
0
x
1
1

Some Common Examples

Determine and plot the curve defined by

$$x = \sec\theta, \quad y = \tan\theta, \quad 0 \le \theta < \frac{\pi}{2}$$

$x^2 = \sec^2\theta \qquad y^2 = \tan^2\theta \qquad \Longrightarrow$

$y^2 + 1 = x^2 \qquad \Longrightarrow \qquad x^2 - y^2 = 1$

hyperbola center $(0,0)$.
open left/right.

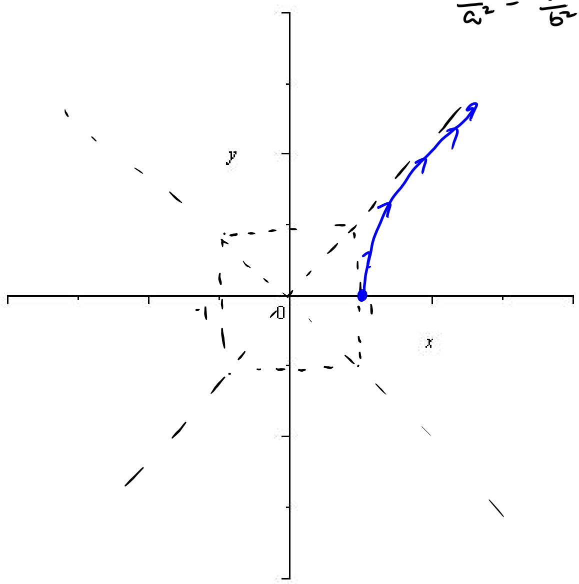
when $\theta = 0$, $\quad x = \sec 0 = 1 \qquad y = \tan 0 = 0$

For $0 \le \theta < \frac{\pi}{2}$, $x > 0$ and $y \ge 0$

$$\lim_{\theta \to \frac{\pi}{2}^-} x = \lim_{\theta \to \frac{\pi}{2}^-} \sec\theta = \infty$$

$$\lim_{\theta \to \frac{\pi}{2}^-} y = \lim_{\theta \to \frac{\pi}{2}^-} \tan\theta = \infty$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Eliminating the Parameter

Determine and plot the parametric curve by eliminating the parameter. Indicate the direction of the curve with arrows.

$$x = e^s - 1, \quad y = e^{2s}, \qquad -\infty < s < \infty$$

$$e^{2s} = y \implies e^s = \sqrt{y} \implies x = \sqrt{y} - 1$$

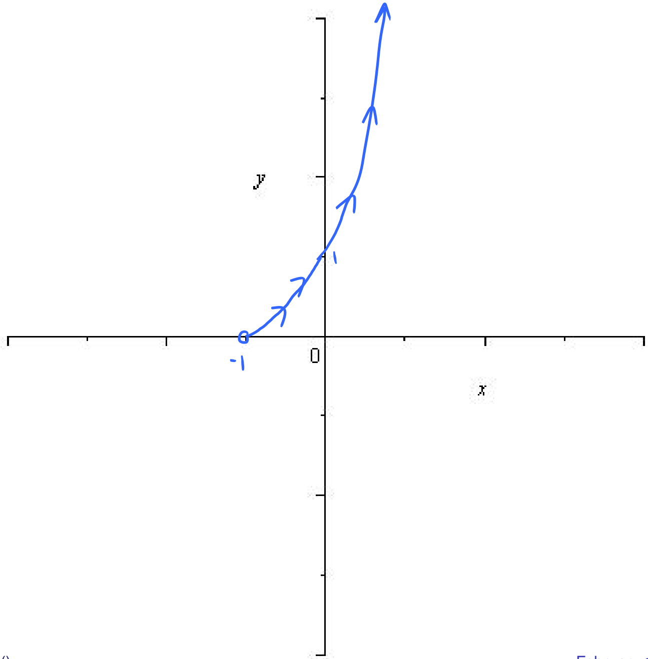
$$x = e^s - 1 \implies e^s = x + 1 \implies y = (x+1)^2$$

parabola open upward

vertex @ $(-1, 0)$

as $s \to -\infty$ $\quad x \to -1$ and $y \to 0$

$s \to \infty$ $\quad x \to \infty$ and $y \to \infty$



y
0
x
-1
1

Finding a Parametrization

Find a set of parametric equations for the line segment from the point (x_0, y_0) to the point (x_1, y_1) with the condition that the parameter $0 \le t \le 1$.

If we take x and y to be lines, then

$$x = at + b \quad \text{and} \quad y = ct + d.$$

when $t = 0$ $\quad x = a \cdot 0 + b = x_0 \implies b = x_0$

$\quad y = c \cdot 0 + d = y_0 \implies d = y_0$

When $t=1$, $x = a \cdot 1 + b = x_1 \Rightarrow$

$$a = x_1 - b = x_1 - x_0$$

Similarly $c = y_1 - d = y_1 - y_0$

Hence
$$\boxed{\begin{aligned} x &= (x_1 - x_0)t + x_0 \\ y &= (y_1 - y_0)t + y_0 \end{aligned}}$$

Section 10.2: Calculus with Parametric Curves

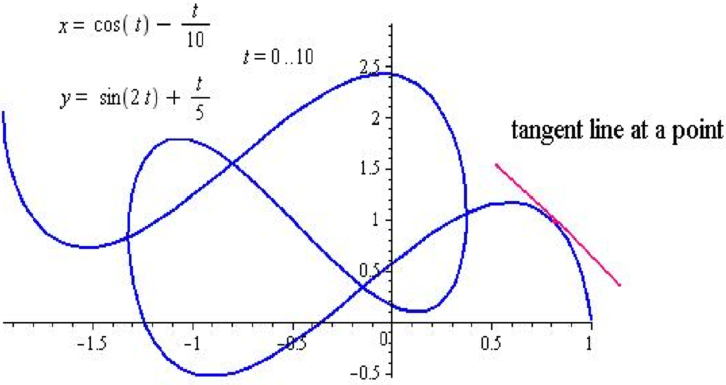


Figure: Even if a parametric curve can not be the graph of a function, we can still ask about the slope of a parametric curve or try to find tangent lines. We need calculus.

Slope of a Parametric Curve

Theorem: Suppose $x = f(t)$ and $y = g(t)$ where f and g are differentiable functions. Then whenever $dx/dt \neq 0$, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} \qquad \text{If } \frac{dx}{dt} \neq 0, \text{ divide}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

Example

Find the equation of the line tangent to the graph of the parametric curve at the indicated parameter value.

$$x = t - \frac{1}{t}, \quad y = 1 + t^2, \quad \text{at} \quad t = 1$$

Need a point and a slope:

point $\quad x(1) = 1 - \frac{1}{1} = 0 \qquad y(1) = 1 + 1^2 = 2$

$(0, 2)$

Slope $\quad m = \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{2t}{1 + 1/t^2}\right|_{t=1} = \frac{2\cdot 1}{1+1} = 1$

$y - 2 = 1(x - 0) \implies y = x + 2$