February 23 Math 2306 sec 59 Spring 2016

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We seek solutions of the form $y = e^{mx}$ for constant m, and obtain the characteristic (a.k.a. auxiliary) equation

$$am^2 + bm + c = 0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ form a fundamental solution set.

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Characteristic eqn $\quad m^2 + m - 12 = 0$

$(m+4)(m-3) = 0 \implies$ $m = -4$ or $m = 3$ 2 distinct real roots

$y_1 = e^{-4x}, \quad y_2 = e^{3x}$

general solution

$$y = c_1 e^{-4x} + c_2 e^{3x}$$

$$y'(x) = -4c_1 e^{-4x} + 3c_2 e^{3x}$$

$$y(0) = c_1 e^0 + c_2 e^0 = 1 \implies c_1 + c_2 = 1$$

$$y'(0) = -4c_1 e^0 + 3c_2 e^0 = 10 \implies -4c_1 + 3c_2 = 10$$

$$\begin{array}{r} 4c_1 + 4c_2 = 4 \\ -4c_1 + 3c_2 = 10 \\ \hline 7c_2 = 14 \end{array} \implies c_2 = 2$$

$$c_1 = 1 - c_2 = 1 - 2 = -1$$

The solution to the IVP is

$$y = -e^{-4x} + 2e^{3x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

$y_2 = y_1 u \qquad u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

Standard form $\quad y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$

$P(x) = \frac{b}{a}, \quad -\int P(x)\,dx = -\int \frac{b}{a}\,dx = -\frac{b}{a} x$

$$u = \int \frac{e^{-\frac{b}{a}x}}{\left(e^{-\frac{b}{2a}x}\right)^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx$$

$$= x \qquad \text{so} \quad y_2 = x\,e^{-\frac{b}{2a}x}$$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic Eqn $\quad 4m^2 - 4m + 1 = 0$

$$(2m-1)^2 = 0$$

$m = \frac{1}{2}$ repeated root

$$y_1 = e^{\frac{1}{2}x}, \quad y_2 = xe^{\frac{1}{2}x}$$

The general solution is

$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}.$$

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Characteristic eqn $\quad m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \implies m = -3$ repeated root

$y_1 = e^{-3x}, \quad y_2 = x e^{-3x}$

General solution $\quad y = c_1 e^{-3x} + c_2 x e^{-3x}$

$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$

$$y(0) = C_1 e^0 + C_2 \cdot 0 e^0 = 4 \quad \Rightarrow \quad C_1 = 4$$

$$y'(0) = -3C_1 e^0 + C_2 e^0 - 3C_2 \cdot 0 e^0 = 0 \qquad -12 + C_2 = 0$$

$$C_2 = 12$$

The solution to the IVP is

$$y = 4e^{-3x} + 12xe^{-3x}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1(x) = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos\beta x + i\sin\beta x\right)$$

$$y_2(x) = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos\beta x - i\sin\beta x\right)$$

$$y_1 = e^{\alpha x}\cos\beta x + i\, e^{\alpha x}\sin\beta x$$

$$y_2 = e^{\alpha x}\cos\beta x - i\, e^{\alpha x}\sin\beta x$$

Let $y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos\beta x\right)$

$$y_1 = e^{\alpha x}\cos\beta x$$

principle of superposition

Let $y_2 = \frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}\left(2i\,e^{\alpha x}\sin\beta x\right)$

$$y_2 = e^{\alpha x}\sin\beta x$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Characteristic eqn $\quad m^2 + 4m + 6 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot 6}}{2} = \frac{-4 \pm \sqrt{-8}}{2} = \frac{-4 \pm 2\sqrt{2}\,i}{2}$$

$m = -2 \pm \sqrt{2}\,i \qquad \alpha = -2 \qquad \beta = \sqrt{2}$

$y_1 = e^{-2x}\cos(\sqrt{2}\,x)$, $\quad y_2 = e^{-2x}\sin(\sqrt{2}\,x)$

General solution

$$y = c_1 e^{-2x}\cos(\sqrt{2}\,x) + c_2 e^{-2x}\sin(\sqrt{2}\,x)$$