February 23 Math 2335 sec 51 Spring 2016

Section 4.1: Polynomial Interpolation

Interpolation is the process of finding a curve or evaluating a function whose curve passes through a known set of points.

A set of points may arise as experimenatal data, discrete measurements of objects (e.g. for computer graphics), or as solutions of a mathematical problem (e.g. numerical differential equations).

We'll consider finding a *nice* function passing through given points—a polynomial.

Linear Interpolation

Given two distinct (i.e. $x_0 \neq x_1$) points (x_0, y_0) and (x_1, y_1), the straight line passing through these points is

$$P_1(x) = \frac{(x_1 - x)y_0 + (x - x_0)y_1}{x_1 - x_0}$$

Evaluate $P_1(x_0)$ and $P_1(x_1)$.

$$P_1(x_0) = \frac{(x_1 - x_0)y_0 + (x_0 - x_0)y_1}{x_1 - x_0} = \frac{(x_1 - x_0)y_0}{x_1 - x_0} = y_0$$

$$P_1(x_1) = \frac{(x_1 - x_1)y_0 + (x_1 - x_0)y_1}{x_1 - x_0} = \frac{(x_1 - x_0)y_1}{x_1 - x_0} = y_1$$

Hence P_1 passes through the pairs

(x_0, y_0) and (x_1, y_1).

Example

Write the equation of the line $P_1(x)$ through $(1,1)$ and $(4,2)$.

$$P_1(x) = \frac{(x_1 - x)\,y_0 + (x - x_0)\,y_1}{x_1 - x_0}$$

Here $x_0 = 1 \quad x_1 = 4 \quad , \quad y_0 = 1 \quad y_1 = 2$

$$P_1(x) = \frac{(4-x)\cdot 1 + (x-1)\cdot 2}{4-1} = \frac{(4-x) + 2(x-1)}{3}$$

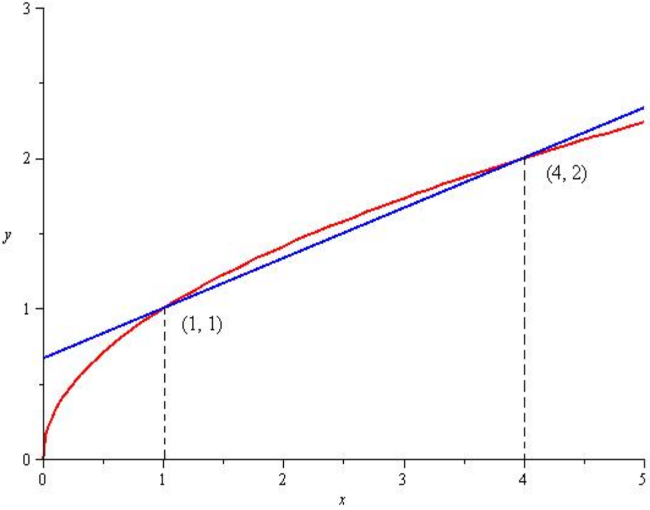

Figure: The curve $f(x) = \sqrt{x}$ together with the linear interpolation $P_1(x)$ through $(1, 1)$ and $(4, 2)$.

Using a Linear Interpolation (example)

Suppose we have a table of values for the tangent function

x	1	1.1	1.2	1.3
$\tan x$	1.5574	1.9648	2.5722	3.6021

Use a linear interpolation to approximate the value tan(1.15).

Let's take $(x_0, y_0) = (1.1, 1.9648)$ $(x_1, y_1) = (1.2, 2.5722)$

$$P_1(x) = \frac{(x_1 - x)\cdot y_0 + (x - x_0)\cdot y_1}{x_1 - x_0} =$$

$$= \frac{(1.2 - x)\,1.9648 + (x - 1.1)\,2.5722}{1.2 - 1.1}$$

Example Continued...[1]

$$P_1(x) = 10\left(1.9648\,(1.2-x) + 2.5722\,(x-1.1)\right)$$

$$= 19.648\,(1.2-x) + 25.722\,(x-1.1)$$

$$\tan(1.15) \approx P_1(1.15) = 19.648\,(1.2-1.15) + 25.722\,(1.15-1.1)$$

$$\doteq 2.2685$$

[1] The true value to four decimal places is $\tan(1.15) = 2.2345$. We will consider error involved in polynomial interpolation in section 4.2

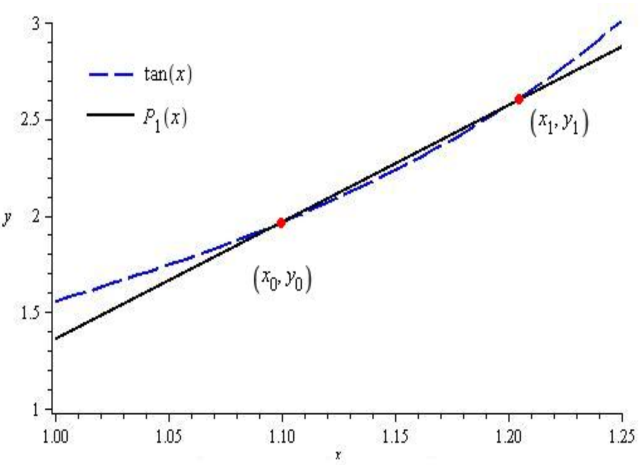

Figure: The curve $f(x) = \tan(x)$ together with the linear interpolation $P_1(x)$ through $(1.1, 1.9648)$ and $(1.2, 2.5722)$. $P_1(1.15) = 2.2685$ so that $\text{Err}(P_1(1.15)) = -0.034$ and $\text{Rel}(P_1(1.15)) = -0.0152$.

Quadratic Interpolation

One weakness of using a linear interpolation is that it can't account for *curviness*. We can stick with using polynomials and allow for a graph that curves by fitting with a quadratic—or higher degree polynomial.

To get a line, we need two distinct points. To get a quadratic, we require three distinct points (x_0, y_0), (x_1, y_1) and (x_2, y_2).

Lagrange Interpolation Basis Functions

We create our polynomial with basic building blocks. These building blocks will be simple polynomials. To motivate, let's look back at the linear interpolation:

Given two points (x_0, y_0) and (x_1, y_1) we had

$$\begin{aligned} P_1(x) &= \frac{(x_1 - x)y_0 + (x - x_0)y_1}{x_1 - x_0} = y_0\left(\frac{x - x_1}{x_0 - x_1}\right) + y_1\left(\frac{x - x_0}{x_1 - x_0}\right) \\ &= y_0 L_0(x) + y_1 L_1(x) \end{aligned}$$

$$\text{Where} \quad L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

Lagrange Interpolating Basis Functions

Consider three different x-values x_0, x_1, and x_2, define three polynomials

$$\begin{aligned}
L_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\
L_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\
L_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}
\end{aligned}$$

These are the *Lagrange interpolating basis functions* for the given x-values.

Lagrange Interpolating Basis Functions

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

Evaluate $L_0(x)$ at each of $x = x_0, x_1$, and x_2.

$$L_0(x_0) = \frac{(x_0 - x_1)(x_0 - x_2)}{(x_0 - x_1)(x_0 - x_2)} = 1$$

$$L_0(x_2) = \frac{(x_2 - x_1)(x_2 - x_2)}{(x_0 - x_1)(x_0 - x_2)} = 0$$

$$L_0(x_1) = \frac{(x_1 - x_1)(x_1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} = 0$$

Lagrange Interpolating Basis Functions

The basis functions have the following property

$$L_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Kronecker Delta Function: is denoted by δ_{ij} (sometimes by δ_i^j) and is defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

So we can write $L_i(x_j) = \delta_{ij}$.

Lagrange's Formula for Interpolating Polynomial

Given three distinct points (x_0, y_0), (x_1, y_1) and (x_2, y_2), the *unique* quadratic polynomial passing through these points is given by

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

where L_0, L_1, and L_2 are the Lagrange basis functions.

This formulation (for P_2) is called the

Lagrange's Formula.

Lagrange's Formula for Interpolating Polynomial

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

Use the property of the Lagrange basis functions to verify that P_2 passes through the three points (x_0, y_0), (x_1, y_1) and (x_2, y_2).

$$P_2(x_0) = y_0 L_0(x_0) + y_1 L_1(x_0) + y_2 L_2(x_0) = y_0 \cdot 1 + y_1 \cdot 0 + y_2 \cdot 0 = y_0$$

$$P_2(x_1) = y_0 L_0(x_1) + y_1 L_1(x_1) + y_2 L_2(x_1) = y_0 \cdot 0 + y_1 \cdot 1 + y_2 \cdot 0 = y_1$$

$$P_2(x_2) = y_0 L_0(x_2) + y_1 L_1(x_2) + y_2 L_2(x_2) = y_0 \cdot 0 + y_1 \cdot 0 + y_2 \cdot 1 = y_2$$

Example

Find the quadratic interpolating polynomial that passes through the points $(0, -1)$, $(1, -1)$, and $(2, 7)$.

x_0 $\quad$ x_1 $\quad$ x_2

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}x(x-1)$$

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$P_2(x) = \frac{-1}{2}(x-1)(x-2) + x(x-2) + \frac{7}{2}x(x-1)$$

This simplifies to

$$P_2(x) = 4x^2 - 4x - 1$$

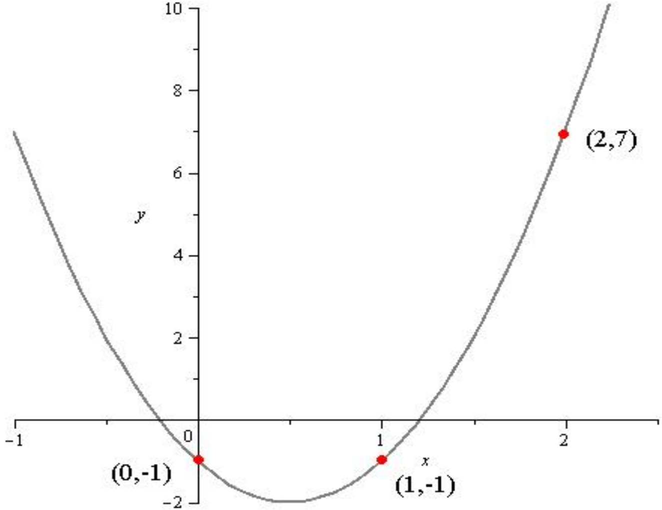

Figure: The points $(0, -1)$, $(1, -1)$, and $(2, 7)$ together with the interpolating polynomial P_2.

Uniqueness of the Interpolating Polynomial

Question: For the three points, could there be two or more quadratics that pass through them? If so, how can we know we've found the *right* one?

Suppose that two quadratics $P_2(x)$ and $Q_2(x)$ both pass through (x_0, y_0), (x_1, y_1) and (x_2, y_2). Determine what must be true about the (at most) quadratic $R(x) = P_2(x) - Q_2(x)$.

$P_2(x) = a_2x^2 + a_1x + a_0$ $\qquad$ $Q_2(x) = b_2x^2 + b_1x + b_0$

$R(x) = (a_2 - b_2)x^2 + (a_1 - b_1)x + (a_0 - b_0)$ $\qquad$ Degree 2 at most!

Look @ $R(x_i)$

$$R(x_0) = P_2(x_0) - Q_2(x_0) = y_0 - y_0 = 0$$

$$R(x_1) = P_2(x_1) - Q_2(x_1) = y_1 - y_1 = 0$$

$$R(x_2) = P_2(x_2) - Q_2(x_2) = y_2 - y_2 = 0$$

A quadratic can't have three real roots.

So $R(x) = 0$ (the zero function).

Thus $P_2(x) = Q_2(x)$.

Using a Quadratic Interpolation (example)

Suppose we have a table of values for the tangent function

		x_0	x_1	x_2
x	1	1.1	1.2	1.3
$\tan x$	1.5574	1.9648	2.5722	3.6021

Use a quadratic interpolation to approximate the value tan(1.15). (Use 1.1, 1.2 and 1.3)

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1.2)(x-1.3)}{(1.1-1.2)(1.1-1.3)} = 50(x-1.2)(x-1.3)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1.1)(x-1.3)}{(1.2-1.1)(1.2-1.3)} = -100(x-1.1)(x-1.3)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1.1)(x-1.2)}{(1.3-1.1)(1.3-1.2)} = 50(x-1.1)(x-1.2)$$

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$= 1.9648(50)(x-1.2)(x-1.3) - 257.22(x-1.1)(x-1.3)$$

$$+ 3.6021(50)(x-1.1)(x-1.2)$$

Example Continued...[2]

$$\tan(1.15) \approx P_2(1.15) \doteq 2.2157$$

[2]Recall that the true value to four decimal places is $\tan(1.15) = 2.2345$.

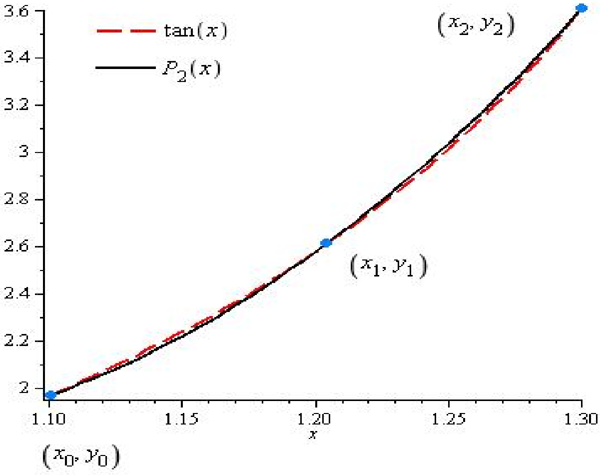

Figure: The curve $f(x) = \tan(x)$ together with the quadratic interpolation $P_2(x)$ through $(1.1, 1.9648)$, $(1.2, 2.5722)$, and $(1.3, 3.6021)$. $P_2(1.15) = 2.2157$ so that $\text{Err}(P_2(1.15)) = 0.0188$ and $\text{Rel}(P_2(1.15)) = 0.0084$.

Higher Degree Interpolation: Lagrange's Formula

Suppose we have $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. We define the $n+1$ Lagrange interpolation basis functions $L_0, L_1, \ldots, L_n$ by

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

for $i = 0, \ldots, n$.

$$L_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$\frac{x-x_j}{x_i-x_j}$$

Lagrange's Formula The unique polynomial of degree $\leq n$ passing through these $n+1$ points is

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x).$$

Example

Find the polynomial of degree at most three that passes through the points $(-1,5)$, $(0,3)$, $(1,1)$, and $(2,11)$.

$x_0 \quad x_1 \quad x_2 \quad x_3$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} = -\frac{1}{6}x(x-1)(x-2)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+1)(x-1)(x-2)}{(1)(0-1)(0-2)} = \frac{1}{2}(x+1)(x-1)(x-2)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)} = \frac{-1}{2}(x+1)x(x-2)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} = \frac{1}{6}(x+1)x(x-1)$$

$$P_3(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$P_3(x) = \frac{-5}{6} x(x-1)(x-2) + \frac{3}{2}(x+1)(x-1)(x-2) - \frac{1}{2}(x+1)x(x-2)$$

$$+ \frac{11}{6}(x+1)x(x-1)$$

This simplifies to

$$P_3(x) = 2x^3 - 4x + 3$$

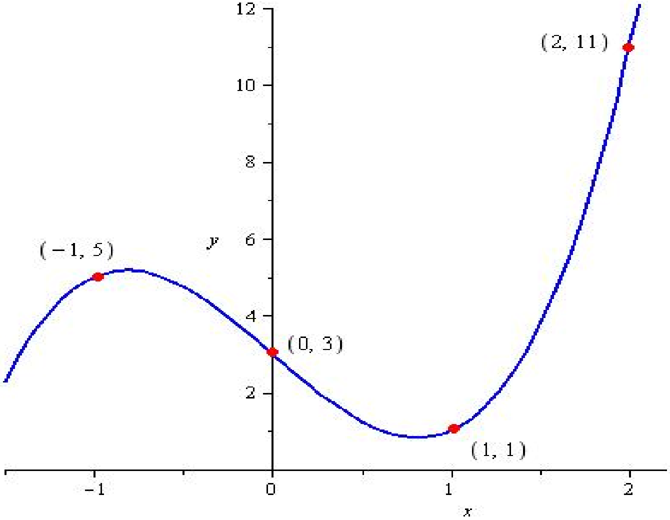

Figure: The points $(-1, 5)$, $(0, 3)$, $(1, 1)$, and $(2, 11)$ together with the interpolating polynomial P_3.

Newton Divided Differences

The quadratic interpolating polynomial for the set of data $(-1,5)$, $(1,1)$, $(2,11)$ is

$$P_2(x) = 4x^2 - 2x - 1.$$

The cubic interpolating polynomial for the set of data $(-1,5)$, $(0,3)$, $(1,1)$, $(2,11)$ is

$$P_3(x) = 2x^3 - 4x + 3.$$

Note that the second set of data is the same as the first with a single additional point included. However, there is no clear connection between the two interpolating polynomials P_2 and P_3 [3].

[3]Both were obtained by using the Lagrange interpolating basis functions from scratch.