Feb. 24 Math 2254H sec 015H Spring 2015

Section 10.2: Calculus with Parametric Curves

Slope of a Parametric Curve

Theorem: Suppose $x = f(t)$ and $y = g(t)$ where f and g are differentiable functions. Then whenever $dx/dt \neq 0$, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Example

Consider the parametric equations

$$x = t^2, \quad y = t^3 - 3t.$$

(a) Determine the point(s) on the curve where the tangent line is horizontal.

(b) Show that the curve has two different tangent lines at the point $(3, 0)$ and find their equations.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} = \frac{3(t^2-1)}{2t}$$

$$\frac{dy}{dx} = 0 \implies t^2 - 1 = 0 \implies t = 1 \text{ or } t = -1$$

when $t=1$, $x=1$ $\quad y = 1-3 = -2$

$t=-1$, $x=1$ $\quad y = -1+3 = 2$

$\boxed{\text{a) } (1,2) \text{ and } (1,-2)}$

$x = t^2 \qquad y = t^3 - 3t$

b) If $x = 3$, $t^2 = 3 \Rightarrow t = \sqrt{3}$ or $t = -\sqrt{3}$

If $y = 0$, $t^3 - 3t = 0 \Rightarrow t(t^2 - 3) = 0$

$t = 0$, $t = \sqrt{3}$ or $t = -\sqrt{3}$

$(x, y) = (3, 0)$ when $t = \sqrt{3}$ and $t = -\sqrt{3}$.

$$\frac{dy}{dx} = \frac{3(t^2 - 1)}{2t}$$

Slopes: $m_1 = \left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{3(3-1)}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$$m_2 = \left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{3(3-1)}{-2\sqrt{3}} = -\sqrt{3}$$

2 lines: $y - 0 = \sqrt{3}\,(x-3)$

$$\boxed{y = \sqrt{3}\,x - 3\sqrt{3}}$$

$$y - 0 = -\sqrt{3}\,(x-3)$$

$$\boxed{y = -\sqrt{3}\,x + 3\sqrt{3}}$$

Graph of $x = t^2$, $y = t^3 - 3t$

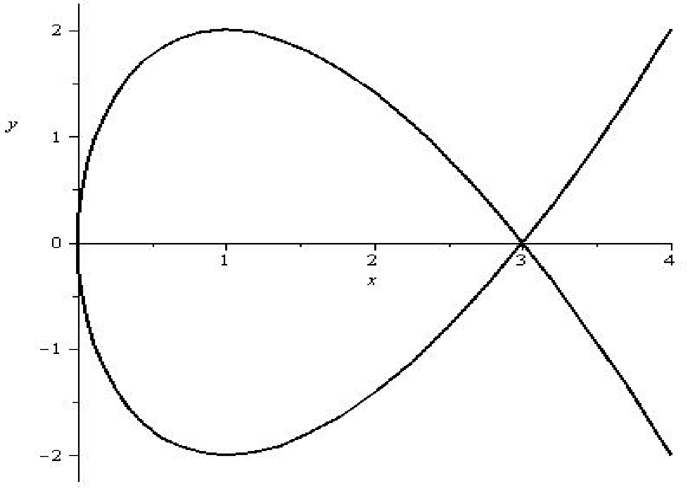


Figure: $x = t^2$, $y = t^3 - 3t$, $-2 \le t \le 2$

The Second Derivative

Observation: Taking a derivative of y with respect to x amounts to taking a derivative of y **with respect to** t, and dividing this by dx/dt.

Theorem: If $x = f(t)$, $y = g(t)$ with f and g sufficiently differentiable and $f'(t) \neq 0$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

CAUTION: $\frac{d^2y}{dx^2}$ is **not** equal to $\frac{d^2y}{dt^2}$ divided by $\frac{d^2x}{dt^2}$.

Example

Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. And find the values of the parameter θ for which the parametric curve would be concave upward.

$$x = \cos 2\theta, \quad y = \cos\theta, \quad 0 < \theta < \pi$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta}{-2\sin 2\theta} = \frac{\sin\theta}{2\sin 2\theta} = \frac{\sin\theta}{4\sin\theta\cos\theta}$$

$$= \frac{1}{4\cos\theta} = \frac{1}{4}\sec\theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{1}{4}\sec\theta\tan\theta}{-2\sin 2\theta}$$

$$= \frac{-1}{8}\,\frac{\sec\theta\tan\theta}{2\sin\theta\cos\theta} = \frac{-1}{16}\,\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta\cos\theta}$$

$$= \frac{-1}{16}\,\frac{1}{\cos^3\theta} = \frac{-1}{16}\sec^3\theta$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-1}{16}\sec^3\theta}$$

The curve is concave up if

$\frac{\pi}{2} < \theta < \pi$ (where $\sec\theta < 0$)

Area Under a Curve

The area bounded between the x-axis and the continuous curve $y = F(x)$ over the interval $[a, b]$ is known to be

$$\text{Area} = \int_a^b |y|\, dx = \int_a^b |F(x)|\, dx.$$

If the curve is traced once by the parametric equations $x = f(t)$, $y = g(t)$ for $\alpha \le t \le \beta$, then by substitution

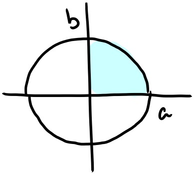
$$\text{Area} = \int_a^b |y|\, dx = \int_\alpha^\beta |g(t)f'(t)|\, dt$$

Example

Find the area enclosed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Note that the ellipse can be parameterized by

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$$

Note $\frac{x}{a} = \cos\theta$, $\frac{y}{b} = \sin\theta$



quarter of ellipse is mapped out for

$$0 \le \theta \le \frac{\pi}{2}$$

$$\frac{1}{4} A = \int_{\theta=0}^{\theta=\pi/2} y\, dx = \int_0^{\pi/2} b\sin\theta \, |-a\sin\theta| \, d\theta$$

$$\Rightarrow \quad A = 4ab \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= 4ab \int_0^{\pi/2} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta \right) d\theta$$

$$= 2ab \int_0^{\pi/2} (1 - \cos 2\theta) \, d\theta$$

$$= 2ab \left[\theta - \tfrac{1}{2}\sin 2\theta \right|_0^{\pi/2}$$

$$= 2ab \left[\frac{\pi}{2} - \tfrac{1}{2}\sin\pi - \left(0 - \tfrac{1}{2}\sin(0)\right) \right]$$

$$\Longrightarrow \quad A = 2ab\left(\frac{\pi}{2}\right) = \pi ab$$