February 24 Math 3260 sec. 55 Spring 2020

Section 3.1: Introduction to Determinants

If A is an $n \times n$ matrix, we defined the determinant of A, denoted $\det(A)$ or $|A|$.

- If $n = 2$, $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$.
- If $n > 2$, letting C_{ij} denote the i, j^{th} cofactor of A

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \text{where } i \text{ is fixed}$$

equivalently

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} \quad \text{where } j \text{ is fixed}$$

A 4×4 Example

Evaluate $\det(A)$ where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$

We can do a cofactor expansion down column 1.

$$\det(A) = \overset{=0}{a_{11}} C_{11} + a_{21} C_{21} + \overset{=0}{a_{31}} C_{31} + a_{41} C_{41}$$

$$= a_{21} C_{21} + a_{41} C_{41}$$

C_{21} and C_{41}

$$C_{21} = (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ -5 & 4 & -2 \end{bmatrix}\right)$$

$$= (-1)^3\left(1\begin{vmatrix} 6 & 2 \\ 4 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 6 \\ -5 & 4 \end{vmatrix}\right)$$

$$= -1\left(-12-8 - 2(-6+10) - (12+30)\right)$$

$$= -(-20-8-42) = 70$$

C_{21} and C_{41}

$$C_{41} = (-1)^{4+1}\det\left(\begin{bmatrix} 1 & 2 & -1 \\ 5 & -7 & 3 \\ 3 & 6 & 2 \end{bmatrix}\right)$$

$$= (-1)^5 \left(1 \begin{vmatrix} -7 & 3 \\ 6 & 2 \end{vmatrix} - 2 \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 5 & -7 \\ 3 & 6 \end{vmatrix} \right)$$

$$= -1 \left(-14 - 18 - 2(10-9) - (30+21) \right)$$

$$= -(-32 - 2 - 51) = 85$$

A 4×4 Example

$$\det\left(\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}\right) = a_{21}C_{21} + a_{41}C_{41}$$

$$= 2(70) + (-2)(85)$$

$$= 2(-15) = -30$$

Triangular Matrices

Definition:

The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$. It is said to be **lower triangular** if $a_{ij} = 0$ for all $j > i$. A matrix that is both upper and lower triangular is **diagonal**.

Theorem: For $n \geq 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries. (i.e. if $A = [a_{ij}]$ is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.)

Example

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

upper triangular

$\det(A) = (-1)(2)(3)(-4)(6)$
$= 144$

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{bmatrix}$$

lower triangular

$\det(A) = 7(6)(2)(2)$
$= 42(4) = 168$

Section 3.2: Properties of Determinants

Theorem: Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[1]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then
$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then
$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then
$$\det(B) = k\det(A).$$

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[1] If "row" is replaced by "column" in any of the operations, the conclusions still follow.

Example: Compute the Determinant

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

call the matrix A

We'll find det(B) where B is upper triangular and obtained by doing row ops to A.

Changes to det

$R_1 \leftrightarrow R_4$ $\quad (-1)$

$$\begin{bmatrix} -2 & -5 & 4 & -2 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -5 & 4 & -2 \\ 0 & 0 & -3 & 1 \\ 0 & 3 & 6 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$R_1 + R_2 \to R_2$

no change

$$\begin{bmatrix} -2 & -5 & 4 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$R_4 \leftrightarrow R_2$

(-1)

$$\begin{bmatrix} -2 & -5 & 4 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

no change

$$\begin{bmatrix} -2 & -5 & 4 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad R_3 \leftrightarrow R_4 \qquad (-1)$$

Calling the new, triangular matrix B,

$$\det(B) = -2(1)(-3)(5) = 30$$

And $\det(B) = (-1)(-1)(-1)\det(A)$

$$\Rightarrow \det(A) = -1\det(B) = -30$$

Some Theorems:

Theorem: The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem: For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem: For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Example

Show that if A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Since A is invertible, $\det(A) \neq 0$.

Note $A^{-1}A = I$ and $\det(I) = (1)^n = 1$

So $\det(A^{-1}A) = \det(I) = 1$

$\det(A^{-1})\det(A) = 1$

Divide by $\det(A)$ to get

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that

$$B = P^{-1}AP.$$

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP)$$

$$= \det(P^{-1})\det(A)\det(P)$$

$$= \det(P^{-1})\det(P)\det(A)$$

$$= \frac{1}{\det(P)}\det(P)\det(A)$$

* multiplication of real numbers commutes

$$= 1 \det(A)$$

$$= \det(A)$$