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Section 4.1: Polynomial Interpolation

Context: We consider a set of distinct data points $\{(x_i, y_i) \,|\, i = 0, \dots, n\}$ that we wish to fit with a polynomial curve.

- For a set of $n+1$ points, we can fit a polynomial $P_n(x)$ of degree at most n.
- We assume that the points are distinct in the sense that $x_i \neq x_j$ when $i \neq j$.
- We will have two formulations, a Lagrange formulation and a Newton divided difference formulation.

Lagrange Interpolation Basis Functions

Linear Case: Given two points (x_0, y_0) and (x_1, y_1) we have

$$P_1(x) = y_0 L_0(x) + y_1 L_1(x)$$

$$\text{Where} \quad L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

The functions L_0 and L_1 are examples of

Lagrange Basis Functions.

Lagrange Interpolating Basis Functions

Quadratic Case: Given three distinct points (x_0, y_0), (x_1, y_1), and (x_2, y_2) define

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

These are the *Lagrange interpolating basis functions* for building a quadratic with the given x-values. The unique interpolating polynomial of degree at most 2 for these points is

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

Higher Degree Interpolation: Lagrange's Formula

Suppose we have $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. We define the $n+1$ Lagrange interpolation basis functions $L_0, L_1, \ldots, L_n$ by

$$L_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

for $i = 0, \ldots, n$.

Compactly: $$L_i(x) = \prod_{k=0, k\neq i}^{n} \left(\frac{x - x_k}{x_i - x_k}\right), \quad i = 0, \ldots, n$$

Lagrange's Formula The unique polynomial of degree $\leq n$ passing through these $n+1$ points is

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x).$$

Sifting Property

The basis functions have the following property

$$L_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Kronecker Delta Function: is denoted by δ_{ij} (sometimes by δ_i^j) and is defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

So we can write $L_i(x_j) = \delta_{ij}$.

Example

Find $P_1(x)$ given the pair of points $(0,1)$ and $(4,5)$.

(x_0, y_0) (x_1, y_1)

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-4}{0-4} = \frac{-1}{4}(x-4)$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-0}{4-0} = \frac{1}{4}x$$

$$P_1(x) = y_0 L_0(x) + y_1 L_1(x) = 1\left[\frac{-1}{4}(x-4)\right] + 5\left[\frac{1}{4}x\right]$$

$$P_1(x) = x + 1$$

Example

Find $P_2(x)$ given the points $(0,1)$, $(4,5)$, and $(2,-1)$.

(x_0, y_0) (x_1, y_1) (x_2, y_2)

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-4)(x-2)}{(0-4)(0-2)} = \frac{1}{8}(x-4)(x-2)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(4-0)(4-2)} = \frac{1}{8}x(x-2)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-4)}{(2-0)(2-4)} = \frac{-1}{4}x(x-4)$$

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$= 1\left[\frac{1}{8}(x-2)(x-4)\right] + 5\left[\frac{1}{8}x(x-2)\right] - 1\left[\frac{-1}{4}x(x-4)\right]$$

$$= \frac{1}{8}\left(x^2 - 6x + 8\right) + \frac{5}{8}\left(x^2 - 2x\right) + \frac{1}{4}\left(x^2 - 4x\right)$$

$$P_2(x) = x^2 - 3x + 1$$

Newton Divided Differences

For the data set $(0, 1)$ and $(4, 5)$, we found that

$$P_1(x) = x + 1.$$

And for the data set $(0, 1)$, $(4, 5)$, and $(2, -1)$, we found that

$$P_2(x) = x^2 - 3x + 1$$

Note: The second set is the same as the first with a single additional point. However, there is no obvious connection between the two polynomials P_1 and P_2.

Newton Divided Differences

We would like an alternative formulation that would allow us to compute P_2 from P_1, or perhaps P_3 from P_2 [1] for a common data set.

Recall: If we have a sufficiently differentiable function f, then the Taylor polynomial

$$\begin{aligned} p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= p_2(x) + \text{an extra term.} \end{aligned}$$

Remark: For a data set, there isn't a known function to take derivatives of. So we need something that *serves the same purpose* as derivatives.

[1] More generally, to compute P_k from P_{k-1}.

Newton Divided Differences

Definition: Let f be a function whose domain contains the two distinct numbers x_0 and x_1. We define the *first-order divided difference* of $f(x)$ by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Notation: We'll use the square brackets "[ ]" with commas between the numbers to denote the divided difference.

Example (Newton Divided Difference)

Compute the first-order divided difference $f[0.1, 0.2]$.

(a) Given $f(x) = 2x^2$

$$f[0.1, 0.2] = \frac{f(0.2) - f(0.1)}{0.2 - 0.1}$$

$$= \frac{2(0.2)^2 - 2(0.1)^2}{0.1} = \frac{0.08 - 0.02}{0.1}$$

$$= \frac{0.06}{0.1} = 0.6$$

Example

Compute the first-order divided difference $g[0.1, 0.2]$.

(b) Given $g(0.1) = -2$ and $g(0.2) = 1.1$

$$g[0.1, 0.2] = \frac{g(0.2) - g(0.1)}{0.2 - 0.1}$$

$$= \frac{1.1 - (-2)}{0.1} = \frac{3.1}{0.1} = 31$$

Some Properties of 1^{st} order Divided Difference

Symmetry: $f[x_0, x_1] = f[x_1, x_0]$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{-\left(f(x_0) - f(x_1)\right)}{-(x_0 - x_1)} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= f[x_1, x_0]$$

Some Properties of 1^{st} order Divided Difference

Relation to Derivative: If f is differentiable on the interval $x_0 \leq x \leq x_1$, then by the Mean Value Theorem there exists a number c between x_0 and x_1 such that

$$f[x_0, x_1] = f'(c).$$

So if x_0 and x_1 are *close together* and f is differentiable, then

$$f[x_0, x_1] \approx f'\left(\frac{x_0 + x_1}{2}\right).$$

Example (of second property)

Given the table of values, approximate the value $\sec^2(1.15)$.

x	1	1.1	1.2	1.3
$\tan x$	1.5574	1.9648	2.5722	3.6021

If $f(x) = \tan x$ then $f'(x) = \sec^2 x$.

$$f[1.1, 1.2] \approx f'\left(\frac{1.1+1.2}{2}\right) = f'(1.15)$$

$$f[1.1, 1.2] = \frac{f(1.2) - f(1.1)}{1.2 - 1.1} = \frac{2.5722 - 1.9648}{0.1}$$

$$= \frac{0.6074}{0.1} = 6.074$$

Example Continued...

Use the value $\sec^2(1.15) = 5.9930$ (which is correct to four decimal places) to determine the error and relative error.

Here $x_A = 6.074$ and $x_T = 5.9930$

$$\text{Err}(x_A) = x_T - x_A = 5.9930 - 6.074 \doteq -0.0810$$

$$\text{Rel}(x_A) = \frac{\text{Err}(x_A)}{x_T} = \frac{-0.0810}{5.9930} \doteq -0.0135$$

Higher Order Divided Differences

Suppose we start with three distinct values x_0, x_1, x_2 in our domain. We can compute two first order divided differences

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{and} \quad f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Definition: The *second-order divided difference* of $f(x)$ at the points $x_0, x_1,$ and x_2 is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

Higher Order Divided Differences

Let $x_0, x_1, \ldots, x_n$ be distinct numbers in the domain of the function f.

Definition: The *third-order divided difference* of $f(x)$ at the points x_0, x_1, x_2, and x_3 is

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}.$$

Definition: The n^{th}*-order divided difference* of $f(x)$ at the points $x_0, \ldots, x_n$ is

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

Properties of Newton Divided Differences

Symmetry: Let $\{x_{i_0}, x_{i_1}, \ldots, x_{i_n}\}$ be any permutation (rearrangement) of the numbers $\{x_0, x_1, \ldots, x_n\}$. Then

$$f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}] = f[x_0, x_1, \ldots, x_n].$$

(That is, the order of the *x*-values doesn't affect the value of the divided difference!)

Example

Consider the set of data

x	0	2	4
$f(x)$	1	-1	5

Compute the second order divided differences $f[0,2,4]$ and $f[4,0,2]$.

$$f[0,2,4] = \frac{f[2,4]-f[0,2]}{4-0}, \quad f[4,0,2] = \frac{f[0,2]-f[4,0]}{2-4}$$

$$f[2,4] = \frac{f(4)-f(2)}{4-2} = \frac{5-(-1)}{2} = 3$$

$$f[0,2] = \frac{f(2)-f(0)}{2-0} = \frac{-1-1}{2} = -1$$

$$f[4,0] = \frac{f(0) - f(4)}{0-4} = \frac{1-5}{-4} = 1$$

$$f[0,2,4] = \frac{3-(-1)}{4-0} = \frac{4}{4} = 1$$

$$f[4,0,2] = \frac{-1-1}{2-4} = \frac{-2}{-2} = 1$$

Properties of Newton Divided Differences

Relation to Derivatives:

Theorem: Suppose f is n times continuously differentiable on an interval $\alpha \leq x \leq \beta$, and that $x_0, \ldots, x_n$ are distinct numbers in this interval. Then

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some number c between the smallest and the largest of the numbers $x_0, \ldots, x_n$.

Example

Given the table of values, approximate the value $\sec^2(1.1)\tan(1.1)$. [2]

x	1	1.1	1.2	1.3
$\tan x$	1.5574	1.9648	2.5722	3.6021

For $f(x) = \tan x$, $\frac{1}{2}f''(1.1) \approx f[1, 1.1, 1.2]$

we need $f[1.1, 1.2] = 6.074$ (from earlier)

and $f[1, 1.1] = \dfrac{1.9648 - 1.5574}{1.1 - 1} = 4.074$

[2] If $f(x) = \tan x$, then $\frac{1}{2}f''(x) = \sec^2(x)\tan(x)$.

$$\sec^2(1.1)\tan(1.1) \approx \frac{f[1.1, 1.2] - f[1, 1.1]}{1.2 - 1.0}$$

$$= \frac{6.074 - 4.074}{0.2} = \frac{2}{0.2} = 10.000$$

Interpolating Polynomial: Newton Divided Difference Formula

Suppose we have $n+1$ distinct data points $(x_0, f(x_0))$, $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$.

Linear Interpolation: The linear interpolating polynomial through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ can be written as

$$P_1(x) = f(x_0) + (x - x_0)f[x_0, x_1].$$

Quadratic Interpolation: The quadratic interpolating polynomial through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ can be written as

$$P_2(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

Important Observation 1

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1].$$

Show that $P_1(x_1) = f(x_1)$.

Recall $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$P_1(x_1) = f(x_0) + (x_1 - x_0)\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)$$

$$= f(x_0) + f(x_1) - f(x_0) = f(x_1)$$

Important Observation 2

$$P_1(x) = f(x_0) + (x - x_0)f[x_0, x_1], \quad \text{and}$$

$$P_2(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

Notice that

$P_2(x) = P_1(x) +$ **an extra term**!

Example

Compute $P_1(x)$ using Newton divided differences with $(x_0, y_0) = (0, 1)$ and $(x_1, y_1) = (4, 5)$.

Need $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{5-1}{4} = 1$

$$P_1(x) = f(x_0) + (x - x_0)\, f[x_0, x_1]$$

$$P_1(x) = 1 + (x - 0) \cdot 1 = x + 1$$

Example Extended...

Now compute $P_2(x)$ if the point $(x_2, y_2) = (2, -1)$ is added to the data.

$$P_2(x) = P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

From before $f[0, 4, 2] = 1$

$$P_2(x) = x + 1 + x(x-4) \cdot 1 = x + 1 + x^2 - 4x$$

$$= x^2 - 3x + 1$$

Interpolating Polynomial: Newton Divided Difference Formula

Higher degree polynomials are defined recursively

Cubic Interpolation: The cubic interpolating polynomial through $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ can be written as

$$P_3(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] +$$

$$+(x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

Note that

$$P_3(x) = P_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

Interpolating Polynomial: Newton Divided Difference Formula

k^{th} **Degree Interpolation:** For $k \geq 2$, the polynomial of degree at most k through the points $(x_0, f(x_0)), \ldots (x_k, f(x_k))$ is

$$P_k(x) = P_{k-1}(x) + (x - x_0)(x - x_1)\cdots(x - x_{k-1})f[x_0, \ldots, x_k]$$