February 26 Math 3260 sec. 55 Spring 2020

Section 3.3: Crammer's Rule, Volume, and Linear Transformations

Crammer's Rule is a method for solving a square system $A\mathbf{x} = \mathbf{b}$ by use of determinants. While it is impractical for large systems, it provides a fast method for some small systems (say 2×2 or 3×3).

Definition: For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1} \ \mathbf{b} \ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Crammer's Rule

Theorem: Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Example

Determine whether Crammer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 9 \\ -x_1 & + & 7x_2 & = & -3 \end{array}$$

In the form $A\vec{x} = \vec{b}$, we have

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 14 - (-1) = 15$

$15 \neq 0 \Rightarrow A$ is non singular

$$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}$$

$$\det(A_1(b)) = 63 - (-3) = 66 \qquad \det(A_2(\vec{b})) = -6 - (-9) = 3$$

$$\det(A) = 15$$

The solution $$x_1 = \frac{\det(A_1(\vec{b}))}{\det A} = \frac{66}{15} = \frac{22}{5}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3}{15} = \frac{1}{5}$$

Application

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Crammer's rule.

$$\begin{aligned} 3sX &- 2Y = 4 \\ -6X &+ sY = 1 \end{aligned}$$

In matrix form

$$\underbrace{\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{\vec{X}} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{b}}$$

The system has a unique solution if $\det(A) \neq 0$

$$\det(A) = 3s^2 - 12 = 3(s^2-4) = 3(s-2)(s+2)$$

There is a unique solution provided

$$s \neq \pm 2$$

$$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

$$\det(A_1(\vec{b})) = 4s + 2 \qquad \det(A_2(\vec{b})) = 3s + 24$$

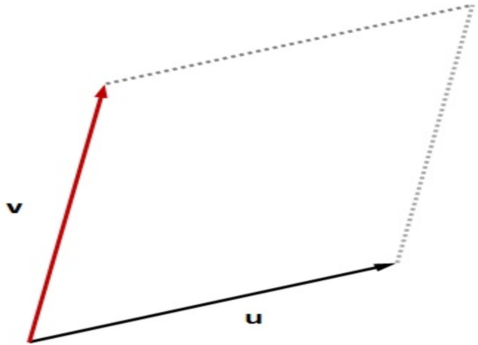
$$\det(A) = 3(s^2 - 4)$$

So the solution for $s \neq \pm 2$ is

$$x = \frac{4s+2}{3(s^2-4)} \quad \text{and}$$

$$y = \frac{3(s+8)}{3(s^2-4)} = \frac{s+8}{s^2-4}$$

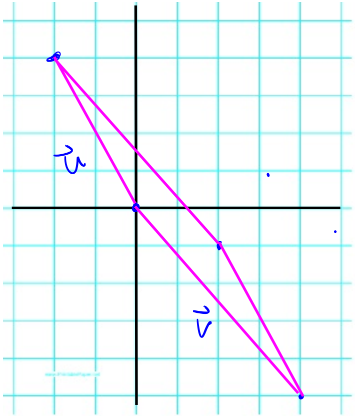
Area of a Parallelogram



Theorem: If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



$\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

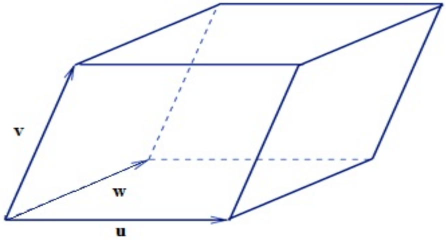
$\det\left(\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\right)$

$= \det \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}$

$= 10 - 16 = -6$

The area is $|-6| = 6$ square units.

Volume of a Parallelopiped

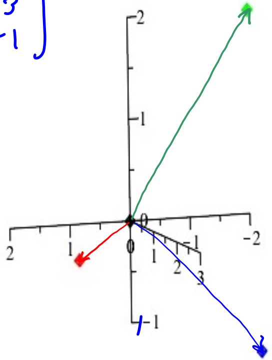


Theorem: If **u**, **v**, and **w** are nonzero, non-collinear vectors in $\mathbb{R}^3$, then the volume of the parallelopiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

Example

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(2, 3, 0)$, $(-2, 0, 2)$ and $(-1, 3, -1)$.

$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$



$\det\left(\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right) \leftarrow A$

$= \det\left(\begin{bmatrix} 2 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}\right)$

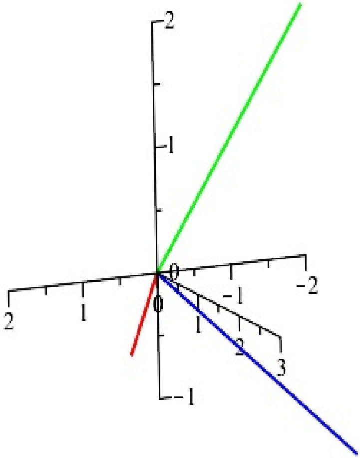
Going down column 1

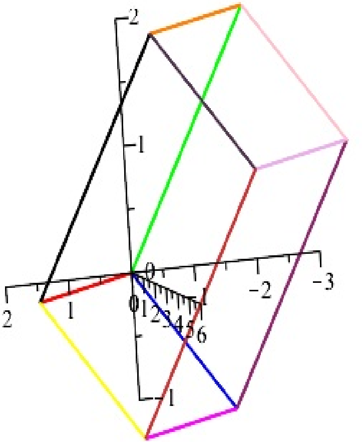
$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + \underbrace{a_{31}}_{=0}C_{31}$$

$$= 2(-1)^{1+1}\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} + 3(-1)^{2+1}\begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix}$$

$$= 2(0-6) - 3(2+2) = -24$$

The volume $V = |-24| = 24$ cubic units.


2
1
0
−1
−2
0
1
2
3
−1


2
1
0
−1
−2
−3
0
1
2
3
4
5
6
−1