Section 10.2: Calculus with Parametric Curves

Length of a Curve If a curve $y = f(x)$ is traced out for $a \le x \le b$, the length of the curve is

$$L = \int_a^b ds \quad \text{where} \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Recall that $ds^2 = dx^2 + dy^2$. Hence we can write ds in terms of dt when $x = f(t)$ and $y = g(t)$.

$$ds^2 = dx^2 + dy^2 = \left[(dx)^2 + (dy)^2\right] \left(\frac{dt}{dt}\right)^2$$

$$ds^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt^2$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

Length of a Parametric Curve

The length of the curve, traversed once, with parametric representation

$$x = f(t), \quad y = g(t), \quad \text{for} \quad \alpha \le t \le \beta$$

is given by

$$L = \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Example

Find the length of the curve with parametric equations

$$x = \frac{t^2}{2} - \ln t, \quad y = 2t + 1, \quad 1 \le t \le 2$$

$x'(t) = t - \frac{1}{t}$ $\qquad$ $y'(t) = 2$

$$(x')^2 + (y')^2 = \left(t - \frac{1}{t}\right)^2 + 2^2$$

$$= t^2 - 2 + \frac{1}{t^2} + 4$$

$$= t^2 + 2 + \frac{1}{t^2} = \left(t + \frac{1}{t}\right)^2$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = \sqrt{\left(t + \frac{1}{t}\right)^2}\, dt$$

$$L = \int_1^2 \sqrt{\left(t+\frac{1}{t}\right)^2}\,dt = \int_1^2 \left|t+\frac{1}{t}\right| dt$$

$$= \int_1^2 \left(t+\frac{1}{t}\right) dt$$

Since $t+\frac{1}{t} > 0$ for $1 \le t \le 2$

$$= \frac{t^2}{2} + \ln|t| \Big|_1^2$$

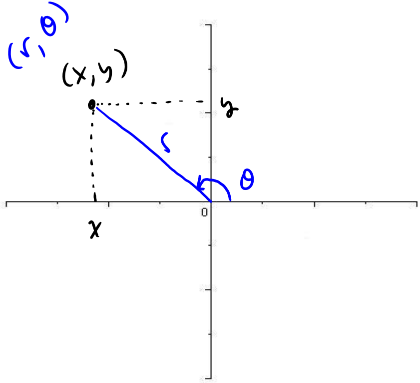
$$= \frac{2^2}{2} + \ln|2| - \left(\frac{1^2}{2} + \ln|1|\right)$$

$$= 2 + \ln 2 - \frac{1}{2} = \frac{3}{2} + \ln 2$$

Section 10.3: Polar Coordinates

Given a point in the plane, we can completely characterize it (relative to an origin) with an ordered pair

$$(x, y).$$



Pole, Polar Axis, Polar Coordinates

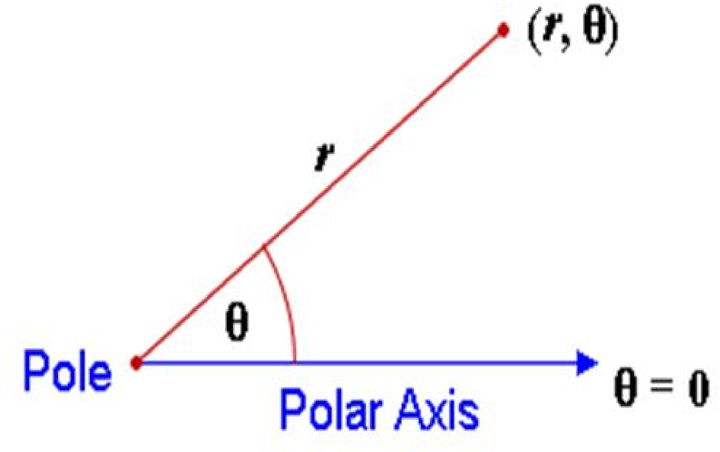
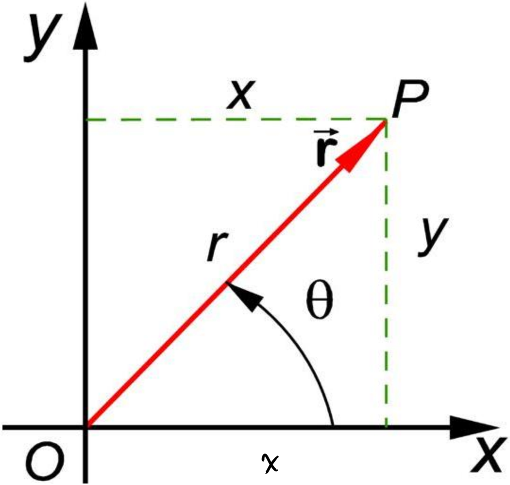


Figure: Note the correct notation is (distance first, angle second).

Connecting Polar to Rectangular Coordinates



$\cos\theta = \frac{x}{r}$

$x = r\cos\theta$

$\sin\theta = \frac{y}{r}$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$

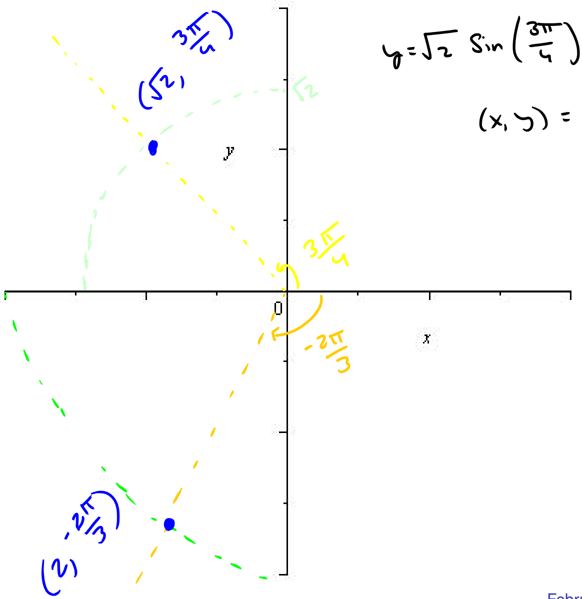
for $x \neq 0$

Plot $(\sqrt{2}, \frac{3\pi}{4})$ and $(2, -\frac{2\pi}{3})$

$x = \sqrt{2}\cos\left(\frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{\sqrt{2}} = -1$

$y = \sqrt{2}\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{\sqrt{2}} = 1$

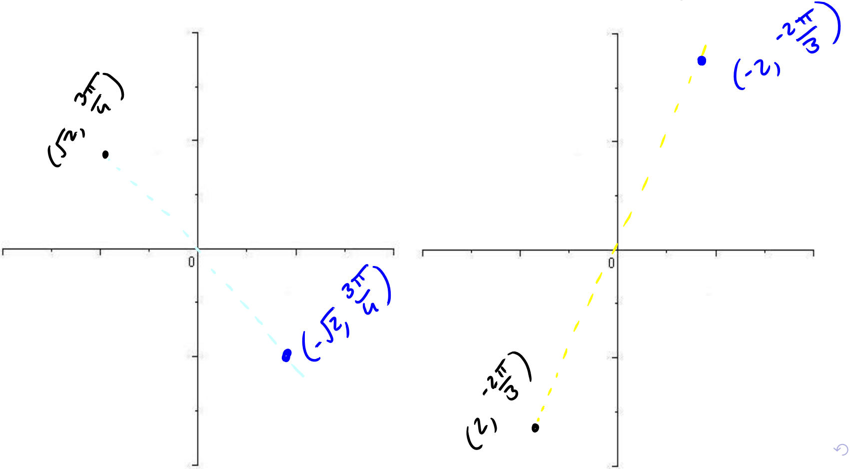
$(x, y) = (-1, 1)$



Negative Values of r

If $r > 0$, then the point $(-r, \theta) = (r, \theta + \pi)$.

Plot the points $(-\sqrt{2}, \frac{3\pi}{4})$, and $(-2, -\frac{2\pi}{3})$.



Converting between Coordinate Systems

The coordinates (x, y) are called **rectangular** or **Cartesian**[1] coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x} \quad \text{for } x \neq 0$$

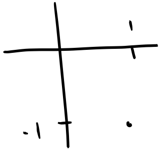
If $x = 0$, then $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$—of course any co-terminal θ may be used.

[1] Named in honor of René Descartes, the *father* of Analytic Geometry.

Examples

Express the Cartesian point in polar coordinates. (Choose $r > 0$.)

(a) $(1, -1)$



quad $\overline{\text{IV}}$

$$r^2 = 1^2 + (-1)^2 = 2$$

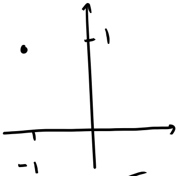
$$r = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 \qquad \tan^{-1}(-1) = \frac{-\pi}{4}$$

$$\theta = \frac{-\pi}{4}$$

$$(r, \theta) = \left(\sqrt{2}, \frac{-\pi}{4}\right)$$

(b) $(-1, 1)$



Quad II

$r^2 = x^2 + y^2 = (-1)^2 + 1^2 = 2$

$r = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1 \qquad \tan^{-1}(-1) = \frac{-\pi}{4}$

$\theta = \tan^{-1}(-1) + \pi = \frac{3\pi}{4}$

$(r, \theta) = \left(\sqrt{2}, \frac{3\pi}{4}\right)$

(c) $(-3\sqrt{3}, -3)$

$$r^2 = x^2 + y^2 = (-3\sqrt{3})^2 + (-3)^2 = 27 + 9$$

$$= 36$$

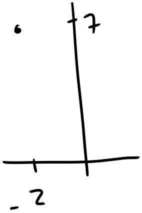
$$r = 6$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \qquad \theta = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$$

$$(r, \theta) = \left(6, \frac{7\pi}{6}\right)$$

(d) $(-2, 7)$



quad II

$$r^2 = (-2)^2 + 7^2 = 4 + 49 = 53$$

$$r = \sqrt{53}$$

$$\tan\theta = \frac{y}{x} = -\frac{7}{2} \qquad \tan^{-1}\left(-\frac{7}{2}\right)$$

$$\theta = \tan^{-1}\left(-\frac{7}{2}\right) + \pi$$

$$(r, \theta) = \left(\sqrt{53},\ \tan^{-1}\left(-\frac{7}{2}\right) + \pi\right)$$

Polar Graphs

Functions in polar coordinates generally appear in the form[2]

$$r = f(\theta).$$

Special Cases: Determine the nature of the graph of the equation in polar coordinates

(a) $r = 2$,

circle of radius 2 centered @ the origin.

$x^2+y^2=r^2 \Rightarrow x^2+y^2=2^2$

(b) $\theta = \dfrac{\pi}{4}$

45° line

$\tan\theta = \dfrac{y}{x}$

$\dfrac{y}{x} = \tan\left(\dfrac{\pi}{4}\right) = 1$

$y = x$

[2]Note the analogy to function of the form $y = f(x)$

Polar Graphs

Evaluate the function expressed in polar coordinates. Plot its graph by converting the equation to Cartesian coordinates.

$$r = 4\sin\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$

$x \neq 0$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

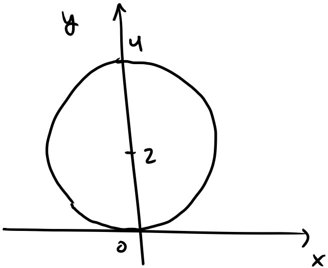
$x^2 + y^2 - 4y = 0$

$x^2 + y^2 - 4y + 4 = 4$

$$x^2 + (y-2)^2 = 2^2$$

cartesian

Circle of radius 2 centered @ $(0,2)$.



Polar Graphs

Converting to Cartesian coordinates isn't always useful (for graphing).
Convert the following to Cartesian coordinates.

$$r = 1 + \cos\theta$$

$$r^2 = r + r\cos\theta$$

$$x^2 + y^2 = \sqrt{x^2+y^2} + x$$

$$x^2 + y^2 - x = \sqrt{x^2+y^2}$$

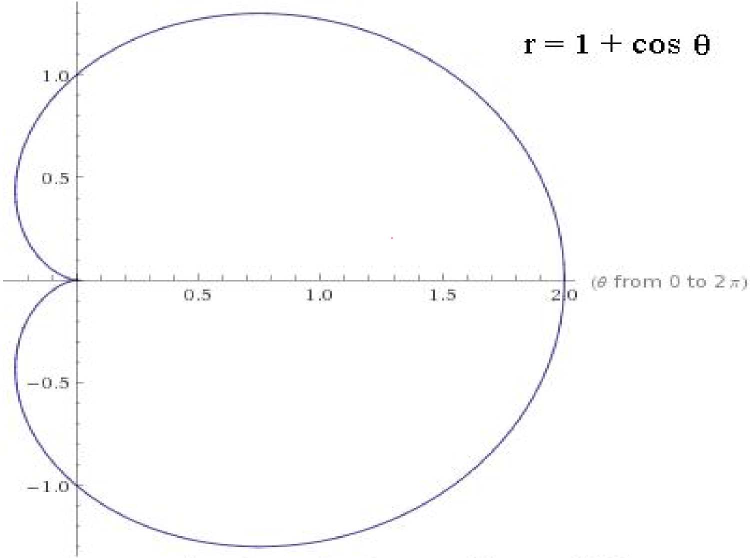
$$(x^2+y^2)^2 - 2x(x^2+y^2) + x^2 = x^2 + y^2$$

$$x^4 + 2x^2y^2 + y^4 - 2x^3 - 2xy^2 + x^2 = x^2 + y^2$$

$$\boxed{x^4 + 2x^2y^2 + y^4 - 2x^3 - 2xy^2 - y^2 = 0}$$

Do we know what the graph should look like based on this ?!?

A plot of the *cardioid* $r = 1 + \cos\theta$



(Produced using Wolfram Alpha)