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Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

We found that $y=e^{mx}$ is a solution provided m is a solution to the equation

$$am^2+bm+c=0$$

called the characteristic (or auxiliary) equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases.

Case I: There are two distinct roots, m_1 and m_2. The two solutions are

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

Case II: There is one repeated real root m. The two solutions are

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

Case III: There is a complex conjugate pair of roots $m = \alpha \pm i\beta$ where α and β are real numbers and $\beta > 0$. The two solutions are

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Characteristic equation: $m^2+4m+6=0$

complete the square $m^2+4m+4-4+6=0$

$$(m+2)^2+2=0$$

$$(m+2)^2=-2$$

$$m+2=\pm\sqrt{-2}=\pm\sqrt{2}\,i$$

$$m=-2\pm\sqrt{2}\,i$$

$m=\alpha\pm i\beta$

$\alpha=-2$ and $\beta=\sqrt{2}$

$$y_1 = e^{\alpha x}\cos(\beta x) \quad , \quad y_2 = e^{\alpha x}\sin(\beta x)$$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t) \quad , \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution is

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \dots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \dots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$y''' - 4y' = 0$

Set $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$, $y''' = m^3 e^{mx}$

$m^3 e^{mx} - 4(me^{mx}) = 0$ $y''' - 4y'$

$e^{mx}(m^3 - 4m) = 0$

$m^3 - 4m = 0$

$m(m^2 - 4) = 0$

$m(m-2)(m+2) = 0$

3 different real roots $m_1 = 0$, $m_2 = 2$, $m_3 = -2$

3 solutions

$$y_1 = e^{0x} = 1\ ,\quad y_2 = e^{2x}\ ,\quad y_3 = e^{-2x}$$

The general solution is

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$$

Example

Solve the ODE

$y''' - 3y'' + 3y' - y = 0$

Characteristic equation $\quad m^3 - 3m^2 + 3m - 1 = 0$

$$(m-1)^3 = 0$$

$m = 1$, repeated root

The solutions are

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x$$

e^{1x}

The general solution is

$$y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e.g. $g(x) = e^{mx}$ for m constant
- sines and/or cosines, e.g. $\sin(kx)$ or $\cos(kx)$ k - constant
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Note: The left is constant coefficient, and the right is a polynomial. Here $g(x) = 8x+1$ which is a 1st degree polynomial. We'll guess that y_p is also a 1st degree polynomial.

Set: $y_p = Ax + B$ for some constants A and B.

$y_p' = A$, $y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 8x+1$$

$$0 - 4(A) + 4(Ax+B) = 8x+1$$

Match coefficients

$$\underline{4A}x + \underline{\underline{(-4A+4B)}} = \underline{8}x + \underline{\underline{1}}$$

This holds if

$$4A = 8$$

$$-4A + 4B = 1$$

Thus $A = 2$, $4B = 1 + 4A = 1 + 2\cdot 4 = 9$

$$B = \frac{9}{4}$$

The particular solution is

$$y_p = 2x + \frac{9}{4}$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Here $g(x) = 6e^{-3x}$ which is a constant times an exponential with -3 in the exponent.

We'll set $y_p = Ae^{-3x}$ for some constant A.

$$y_p' = -3Ae^{-3x}, \quad y_p'' = 9Ae^{-3x}$$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

Matching gives $25A = 6$

$$A = \frac{6}{25}$$

So $y_p = \frac{6}{25}e^{-3x}$