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Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We found that $y = e^{mx}$ is a solution provided m is a solution to the equation

$$am^2 + bm + c = 0$$

called the characteristic (or auxiliary) equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases.

Case I: There are two distinct roots, m_1 and m_2. The two solutions are

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

Case II: There is one repeated real root m. The two solutions are

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

Case III: There is a complex conjugate pair of roots $m = \alpha \pm i\beta$ where α and β are real numbers and $\beta > 0$. The two solutions are

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Characteristic eqn : $m^2 + 4m + 6 = 0$

Let's complete the square $m^2 + 4m + 4 - 4 + 6 = 0$

$$(m+2)^2 + 2 = 0$$

$$(m+2)^2 = -2$$

$$m + 2 = \pm\sqrt{-2} = \pm\sqrt{2}\, i$$

$$m = -2 \pm \sqrt{2}\, i$$

$m = \alpha \pm \beta i$ case $\alpha = -2$ and $\beta = \sqrt{2}$

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t), \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution is

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.
- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \ldots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$y''' - 4y' = 0$

$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}, \quad y''' = m^3 e^{mx}$

$m^3 e^{mx} - 4m e^{mx} = 0 \qquad y''' - 4y' = 0$

$e^{mx}(m^3 - 4m) = 0$

we have a solution provided $m^3 - 4m = 0$

Find the roots $\quad m(m^2-4) = 0$

$m(m-2)(m+2) = 0$

3 distinct real roots $\quad m_1 = 0, \ m_2 = 2, \ m_3 = -2$

3 solutions $\quad y_1 = e^{0x} = 1 \quad y_2 = e^{2x} \quad , y_3 = e^{-2x}$

The general solution is

$$y = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Example

Solve the ODE

$$y''' - 3y'' + 3y' - y = 0$$

Characteristic equation

$$m^3 - 3m^2 + 3m - 1 = 0$$

perfect cube $\quad (m-1)^3 = 0$

$m = 1$ repeated w/ multiplicity 3

we get three linearly independent solutions

$$y_1 = e^x \ , \quad y_2 = x e^x \ , \quad y_3 = x^2 e^x$$

e^{1x}

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{mx} for constant m
- sines and/or cosines, $\sin(kx)$ or $\cos(kx)$
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

$g(x) = 8x+1$ a 1st degree polynomial. (Note the left side is constant coefficient.).

we guess that y_p is a 1st degree polynomial like g. Let's set

$y_p = Ax + B$ for constants A and B.

Substitute into the ODE

$y_p' = A$ and $y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax+B) = 8x + 1$$

We match coefficients to find A and B.

$$\underline{\underline{4A}}\,x + (\underline{\underline{-4A+4B}}) = \underline{\underline{8}}\,x + \underline{\underline{1}}$$

This requires

$$4A = 8 \quad \text{and}$$
$$-4A + 4B = 1$$

$$4A = 8 \implies A = 2, \quad 4B = 1 + 4A = 1 + 4(2) = 9$$

$$B = \frac{9}{4}$$

We found our y_p

$$\boxed{y_p = 2x + \frac{9}{4}}$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

$g(x) = 6e^{-3x}$ is an exponential with $-3x$ in the exponent. We're going to leave the -3 in the exponent. We'll set

$y_p = Ae^{-3x}$ where A is our undetermined coefficient.

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$

$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$

$$9Ae^{-3x} + 12Ae^{-3x} + 4Ae^{-3x} = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

This is true if $A = \frac{6}{25}$

The particular solution is

$$y_p = \frac{6}{25} e^{-3x}$$