February 27 Math 3260 sec. 55 Spring 2018

Section 3.1: Introduction to Determinants

We sought a defiinition of the determinant of a 3×3 matrix with the property that the determinant being zero or nonzero corrolated to the matrix being singular or nonsingular. For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

we defined

$$\text{Det}(A) = a_{11}\text{det}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\text{det}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Definitions: Minors and Cofactors

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

Definition: The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

Definition: Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Observation:

Comparison with the determinant of the 3×3 matrix A, we found a connection to cofactors

$$\mathrm{Det}(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}\end{aligned}$$

(Well call such a sum a **cofactor expansion.**)

Example: Evaluate the determinant

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = 6$

$C_{12} = (-1)^{1+2}M_{12} = (-1)^3 \begin{vmatrix} -2 & 2 \\ 3 & 6 \end{vmatrix} = 18$

$C_{13} = (-1)^{1+3}M_{13} = (-1)^4 \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$

$\det(A) = -1 \cdot 6 + 3(18) + 0(-3) = 48$

Theorem:

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

Example: Find the determinant of the matrix

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix}$$

We can use the zeros in row 2.

A cofactor expansion across row 2

$$\det(A) = \underbrace{a_{21}}_{=0} C_{21} + \underbrace{a_{22}}_{=0} C_{22} + a_{23} C_{23} + \underbrace{a_{24}}_{=0} C_{24}$$

$$C_{23} = (-1)^{2+3} M_{23} = - \begin{vmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{vmatrix} = -48$$ (from previous slide)

$\det(A) = -3(-48) = 144$

We could compute this taking an expansion down column 4

$$\det(A) = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

Triangular Matrices

Definition:

The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$. It is said to be **lower triangular** if $a_{ij} = 0$ for all $j > i$. A matrix that is both upper and lower triangular is **diagonal**.

Theorem: For $n \geq 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries. (i.e. if $A = [a_{ij}]$ is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.)

Example: Evaluate the determinant of each matrix.

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$\det(A) = -1(2)(3)(-4)(6) = 144$

Upper triangular

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{bmatrix}$$

$\det(A) = 7(6)(2)(2) = 168$

Lower triangular

Section 3.2: Properties of Determinants

Theorem: Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[1]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then

$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then

$$\det(B) = k\det(A).$$

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[1] If ”row” is replaced by ”column” in any of the operations, the conclusions still follow.

Example: Compute the Determinant

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

We'll do row operations to get an echelon form (upper triangular) and keep track of any potential changes to the determinant.

Call the original matrix A.

Do $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$

Changes

1. swap (-1)
2. replace no change
3. replace no change
4. swap (-1)

$R_1 + R_4 \to R_4$

(5) scale $\left(\frac{1}{5}\right)$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\frac{1}{5}R_4 \to R_4$$

$$B = \begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(B) = 2 \cdot (1) \cdot (-3) \cdot 1$$

$$= -6$$

Going back to det(A)

$$\det(B) = (-1)\cdot(-1)\cdot\frac{1}{5}\det(A)$$

$$\det(A) = 5(-1)^2\det(B) = 5(1)(-6) = -30$$

Some Theorems:

Theorem: The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem: For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem: For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Example

Show that if A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Note $A^{-1}A = I$ $\qquad \det(I) = 1$

$$\det(A^{-1}A) = \det(I)$$

$$\underbrace{\det(A^{-1})\det(A)}_{\text{scalars}} = 1 \qquad \det(A) \neq 0$$

So $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that

$$B = P^{-1}AP.$$

← called a similarity transformation

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(AP)$$

$$= \underbrace{\det(P^{-1})\det(A)\det(P)}_{\text{real numbers}}$$

$$= \det(A)\underbrace{\det(P^{-1})\det(P)}_{\text{"1"}}$$

$$\det(P^{-1}) = \frac{1}{\det(P)}$$

$$= \det(A)$$