February 28 Math 1190 sec. 62 & 63 Spring 2017

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
  usually x
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.

$$\sin(x+y) = y^2$$

$$\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y^2)$$

derivative

$$\cos(x+y)\cdot\left(1+1\cdot\frac{dy}{dx}\right) = 2y\cdot\frac{dy}{dx}$$

distribute

$$\cos(x+y) + \cos(x+y)\cdot\frac{dy}{dx} = 2y\frac{dy}{dx}$$

sort $\frac{dy}{dx}$ +/-

$$\cos(x+y)\frac{dy}{dx} - 2y\frac{dy}{dx} = -\cos(x+y)$$

factor out $\frac{dy}{dx}$

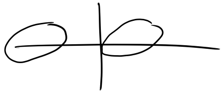
$$\frac{\frac{dy}{dx}(\cos(x+y) - 2y)}{\cos(x+y)-2y} = \frac{-\cos(x+y)}{\cos(x+y)-2y}$$

$$\frac{dy}{dx} = \frac{-\cos(x+y)}{\cos(x+y) - 2y}$$

Example

Find the equation of the line tangent to the graph of the relation $\sin(x+y) = y^2$ at the point $(\pi, 0)$.

$$\frac{dy}{dx} = \frac{-\cos(x+y)}{\cos(x+y) - 2y}$$



point: $(\pi, 0)$

slope $\rightarrow$ $\frac{dy}{dx}$ at $(\pi, 0)$

$$\left.\frac{dy}{dx}\right|_{(\pi,0)} = \frac{-\cos(\pi+0)}{\cos(\pi+0) - 2(0)} = \frac{-(-1)}{-1-0} = -1 \rightarrow \text{slope}$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x - \pi)$$

$$\boxed{y = -x + \pi}$$

Question

Determine $\frac{dy}{dx}$ if $x^2y^3 = 2x - y^2$.

(a) $\frac{dy}{dx} = \frac{2}{3x^2y^2 + 2y}$

(b) $\frac{dy}{dx} = \frac{2 - 2xy^3}{3x^2y^2 + 2y}$

(c) $\frac{dy}{dx} = \frac{2 - 2xy^2}{3x^2y + 2}$

$$\frac{d}{dx}(x^2 \cdot y^3) = \frac{d}{dx}(2x - y^2)$$

$$2xy^3 + x^2 \cdot 3y^2 \cdot \frac{dy}{dx} = 2 - 2y\frac{dy}{dx}$$

$$3x^2y^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2 - 2xy^3$$

$$\frac{dy}{dx}(3x^2y^2 + 2y) = 2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{2 - 2xy^3}{3x^2y^2 + 2y}$$

The Power Rule: Rational Exponents

$\frac{d}{dx}(x^{-3}) = -3x^{-4}$

Let $\boxed{y = x^{p/q}}$ where p and q are integers. This can be written implicitly as

$y = x^{3/2}$

$(y)^q = (x^{\frac{p}{q}})^q$

$$y^q = x^p.$$

Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p)$$

$$\frac{qy^{q-1}\frac{dy}{dx}}{qy^{q-1}} = \frac{px^{p-1}}{qy^{q-1}}$$

$$\frac{dy}{dx} = \frac{p}{q} \cdot \frac{x^{p-1}}{\boxed{y}^{q-1}}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\frac{dy}{dx} = \frac{p}{q} \cdot \frac{x^{p-1}}{\left(x^{p/q}\right)^{q-1}}$$

$(p/q)\cdot(q-1)$

$$= \frac{p}{q} \cdot \boxed{\frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}}}$$

$$x^{(p-1) - \left(\frac{p(q-1)}{q}\right)}$$

$$x^{\frac{q(p-1)}{q} - \frac{p(q-1)}{q}}$$

$$x^{\frac{qp-q}{q} - \frac{pq-p}{q}} = x^{\frac{-q+p}{q}} = x^{-\frac{q}{q}+\frac{p}{q}}$$

$$= \boxed{x^{\frac{p}{q}-1}}$$

$$\frac{dy}{dx} = \frac{p}{q} x^{\frac{p}{q}-1}$$

$$y = x^{\frac{p}{q}}$$

The Power Rule: Rational Exponents

$\frac{1}{4}$
$\downarrow$

Theorem: If r is any rational number, then when x^r is defined, the function $y = x^r$ is differentiable and

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all x such that x^{r-1} is defined.

Examples

Evaluate

(a) $\frac{d}{dx}\sqrt[4]{x} = \frac{d}{dx}\left(x^{\frac{1}{4}}\right) = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$

(b) $\frac{d}{dv}\csc(\sqrt{v}) = \frac{d}{dv}\left(\csc\left(v^{\frac{1}{2}}\right)\right)$

$$= -\csc\left(v^{\frac{1}{2}}\right)\cot\left(v^{\frac{1}{2}}\right)\cdot\frac{d}{dv}\left(v^{\frac{1}{2}}\right)$$

$$= -\csc\left(v^{\frac{1}{2}}\right)\cot\left(v^{\frac{1}{2}}\right)\cdot\frac{1}{2}v^{-\frac{1}{2}}$$

$$= -\frac{\csc(\sqrt{v})\cot(\sqrt{v})}{2\sqrt{v}}$$

Question

Find $g'(t)$ where $g(t) = \sqrt[5]{t}$.

$g(t) = t^{\frac{1}{5}}$

$g'(t) = \frac{1}{5} t^{-\frac{4}{5}}$

$= \frac{1}{5t^{4/5}}$

$= \frac{1}{5\sqrt[5]{t^4}}$

(a) $g'(t) = 5t^4$ left

(b) $g'(t) = -5t^{-6}$ right

(c) $g'(t) = \dfrac{1}{5t^{4/5}}$ both

(d) $g'(t) = \dfrac{1}{5}t^{4/5}$ neither

Inverse Functions

Suppose $y = f(x)$ and $x = g(y)$ are inverse functions—i.e. $(g \circ f)(x) = g(f(x)) = x$ for all x in the domain of f.

Theorem: Let f be differentiable on an open interval containing the number x_0. If $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$. Moreover

$$\frac{d}{dy}g(y_0) = g'(y_0) = \frac{1}{f'(x_0)}.$$

$(x_0, f(x_0))$

Note that this refers to a pair (x_0, y_0) on the graph of f—i.e. (y_0, x_0) on the graph of g. The slope of the curve of f at this point is the reciprocal of the slope of the curve of g at the associated point.

Example

The function $f(x) = x^7 + x + 1$ has an inverse function g. Determine $g'(3)$.

Have a point on the graph of f: $(x, 3)$

$f(x) = 3 = x^7 + x + 1$

$x = 1$ $\qquad (1, 3)$

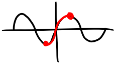
$$g'(3) = \frac{1}{f'(1)}$$

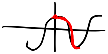
$f'(x) = 7x^6 + 1$

$f'(1) = 7(1)^6 + 1 = 8$

$$g'(3) = \frac{1}{8}$$

Inverse Trigonometric Functions





Recall the definitions of the inverse trigonometric functions.

$y = \arcsin x$

$$y = \sin^{-1} x \quad \iff \quad x = \sin y, \quad -1 \le x \le 1, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

NOT $\frac{1}{\sin x}$

$$y = \cos^{-1} x \quad \iff \quad x = \cos y, \quad -1 \le x \le 1, \quad 0 \le y \le \pi$$

$$y = \tan^{-1} x \quad \iff \quad x = \tan y, \quad -\infty < x < \infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Inverse Trigonometric Functions

There are different conventions used for the ranges of the remaining functions. Sullivan and Miranda use

$$y = \cot^{-1} x \quad \Longleftrightarrow \quad x = \cot y, \quad -\infty < x < \infty, \quad 0 < y < \pi$$

$$y = \csc^{-1} x \quad \Longleftrightarrow \quad x = \csc y, \quad |x| \geq 1, \quad y \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \Longleftrightarrow \quad x = \sec y, \quad |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Derivative of the Inverse Sine

Use implicit differentiation to find $\frac{d}{dx}\sin^{-1}x$, and determine the interval over which $y = \sin^{-1}x$ is differentiable.

$\boxed{y = \sin^{-1}x}$, find $\frac{dy}{dx}$

$\sin y = x$

$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

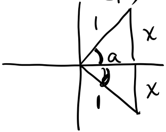
$\frac{\cos y \cdot \frac{dy}{dx}}{\cos y} = \frac{1}{\cos y}$

simplify

$\frac{dy}{dx} = \frac{1}{\cos \boxed{y}} = \boxed{\frac{1}{\boxed{\cos(\sin^{-1}x)}}}$

$$\frac{dy}{dx} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

$\sin^{-1}\left(\frac{x}{1}\right)$



$a^2 + x^2 = 1^2$

$a^2 = 1 - x^2$

$a = \pm\sqrt{1-x^2}$

$(a > 0)$

$a = \sqrt{1-x^2}$

$$\cos\left(\sin^{-1}\left(\tfrac{x}{1}\right)\right) = \frac{\sqrt{1-x^2}}{1} = \boxed{\sqrt{1-x^2}}$$

Where is $\sin^{-1} x$ differentiable?

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

Domain of $\frac{1}{\sqrt{1-x^2}}$: can't have $\sqrt{1-x^2} = 0$

$1 - x^2 = 0$

$x = \pm 1$

Need $1-x^2>0$

$1>x^2$

$$\boxed{-1<x<1}$$

Examples

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Evaluate each derivative

(a) $\displaystyle \frac{d}{dx}\sin^{-1}(e^x) = \frac{1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x)$

$$= \frac{1}{\sqrt{1-(e^x)^2}} \cdot \frac{e^x}{1}$$

$$= \frac{e^x}{\sqrt{1-e^{2x}}}$$

Examples

Evaluate each derivative

(b) $\frac{d}{dx}\left(\sin^{-1}x\right)^3$

outside: $(\quad)^3$

inside: $\sin^{-1}x$

$$= 3\left(\sin^{-1}x\right)^2 \cdot \frac{d}{dx}\left(\sin^{-1}x\right)$$

$$= 3\left(\sin^{-1}x\right)^2 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{3\left(\sin^{-1}x\right)^2}{\sqrt{1-x^2}}$$

Derivative of the Inverse Tangent

Theorem: If $f(x) = \tan^{-1} x$, then f is differentiable for all real x and

$$f'(x) = \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}.$$

The argument uses implicit differentiation just like we used for the inverse sine function. **It is left as an EXERCISE.**

Question: $\frac{d}{du}\tan^{-1} u = \frac{1}{1+u^2}$

Find $\frac{dy}{dx}$ where $y = \tan^{-1}(e^x)$.

(a) $\displaystyle \frac{dy}{dx} = \frac{e^x}{1+e^{2x}}$ left

(b) $\displaystyle \frac{dy}{dx} = \frac{e^x}{1+x^2}$ right

(c) $\displaystyle \frac{dy}{dx} = \frac{1}{1+e^{2x}}$ both

(d) $\displaystyle \frac{dy}{dx} = \sec^{-2}(e^x)$ neither

Derivative of the Inverse Secant

Theorem: If $f(x) = \sec^{-1} x$, then f is differentiable for all $|x| > 1$ and

$$f'(x) = \frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}.$$

Examples

$$\frac{d}{du}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}} \qquad \frac{d}{du}\tan^{-1}u = \frac{1}{1+u^2}$$

Evaluate

(a) $\dfrac{d}{dx}\sec^{-1}(x^2)$

$$= \frac{1}{(x^2)\sqrt{(x^2)^2-1}} \cdot \frac{d}{dx}(x^2) = \frac{2\cancel{x}}{x^{\cancel{2}}\sqrt{x^4-1}} = \frac{2}{x\sqrt{x^4-1}}$$

(b) $\dfrac{d}{dx}\tan^{-1}(\sec x)$

$$= \frac{1}{1+(\sec x)^2} \cdot \frac{d}{dx}(\sec x) = \frac{\sec x \tan x}{1+\sec^2 x}$$

The Remaining Inverse Functions

Due to the trigonometric cofunction identities, it can be shown that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

and

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$