February 28 Math 1190 sec. 62 & 63 Spring 2017

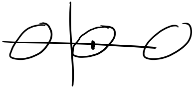
Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
  x
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.



$$\sin(x+y) = y^2$$

$$\frac{d}{dx}\left(\sin(x+y)\right) = \frac{d}{dx}\left(y^2\right)$$

derivative

$$\cos(x+y)\cdot\left(1 + 1\cdot\frac{dy}{dx}\right) = 2y\cdot\frac{dy}{dx}$$

distribute

$$\cos(x+y) + \cos(x+y)\cdot\frac{dy}{dx} = 2y\frac{dy}{dx}$$

sort (+/-)

$$\cos(x+y)\frac{dy}{dx} - 2y\frac{dy}{dx} = -\cos(x+y)$$

factor

$$\frac{\frac{dy}{dx}\left(\cos(x+y) - 2y\right)}{\cos(x+y)-2y} = \frac{-\cos(x+y)}{\cos(x+y)-2y}$$

$$\frac{dy}{dx} = \frac{-\cos(x+y)}{\cos(x+y) - 2y}$$

$$y = x^2 + 3x$$

$$y' = 2x + 3$$

Example

Find the equation of the line tangent to the graph of the relation $\sin(x+y) = y^2$ at the point $(\pi, 0)$.

$$\frac{dy}{dx} = \frac{-\cos(x+y)}{\cos(x+y) - 2y}$$

point: $(\pi, 0)$

Slope $\rightarrow \frac{dy}{dx}$, evaluate at $x = \pi$, $y = 0$

$$\left.\frac{dy}{dx}\right|_{(\pi,0)} = \frac{-\cos(\pi+0)}{\cos(\pi+0) - 2(0)} = \frac{-(-1)}{(-1) - 0} = -1 \rightarrow \text{slope}$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x - \pi)$$

$$y = -x + \pi$$

Question

Determine $\frac{dy}{dx}$ if $x^2y^3 = 2x - y^2$.

$$\frac{d}{dx}(\overset{\overset{f}{\downarrow}}{x^2}\cdot\overset{\overset{g}{\downarrow}}{y^3}) = \frac{d}{dx}(2x-y^2)$$

$$2x\cdot y^3 + x^2\cdot 3y^2\frac{dy}{dx} = 2 - 2y\frac{dy}{dx}$$

$$3x^2y^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2 - 2xy^3$$

$$\frac{dy}{dx}(3x^2y^2+2y) = 2-2xy^3$$

$$\frac{dy}{dx} = \frac{2-2xy^3}{3x^2y^2+2y}$$

(a) $\frac{dy}{dx} = \frac{2}{3x^2y^2+2y}$ left

(b) $\frac{dy}{dx} = \frac{2-2xy^3}{3x^2y^2+2y}$ right

(c) $\frac{dy}{dx} = \frac{2-2xy^2}{3x^2y+2}$ neither

The Power Rule: Rational Exponents

$\frac{d}{dx}(x^{-3}) = -3x^{-4}$

$y = x^{3/2}$

Let $\boxed{y = x^{p/q}}$ where p and q are integers. This can be written implicitly as

$(y)^q = (x^{\frac{p}{q}})^q$

$$y^q = x^p.$$

$(x^a)^b = x^{ab}$

Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p)$$

$$\frac{qy^{q-1}\frac{dy}{dx}}{qy^{q-1}} = \frac{px^{p-1}}{qy^{q-1}}$$

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{\boxed{y}^{q-1}} = \frac{p}{q}\cdot\frac{x^{p-1}}{(x^{p/q})^{q-1}}.$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\frac{dy}{dx} = \frac{p}{q} x^{\frac{p}{q}-1}$$

$$\frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}}$$

$$\frac{x^{p-1}}{x^{\frac{pq-p}{q}}} = x^{p-1-\frac{pq-p}{q}}$$

$$= x^{\frac{q(p-1)}{q} - \frac{pq-p}{q}}$$

$$= x^{\frac{qp-q-pq+p}{q}}$$

$$= x^{\frac{-q+p}{q}}$$

$$= x^{\frac{-q}{q}+\frac{p}{q}} = x^{\frac{p}{q}-1}$$

The Power Rule: Rational Exponents

Theorem: If r is any rational number, then when x^r is defined, the function $y = x^r$ is differentiable and

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all x such that x^{r-1} is defined.

Examples

Evaluate

(a) $\frac{d}{dx}\sqrt[4]{x} = \frac{d}{dx}\left(x^{\frac{1}{4}}\right) = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$

(b) $\frac{d}{dv}\csc(\sqrt{v}) = \frac{d}{dv}\csc\left(v^{\frac{1}{2}}\right) = -\csc\left(v^{\frac{1}{2}}\right)\cot\left(v^{\frac{1}{2}}\right)\cdot\frac{d}{dv}\left(v^{\frac{1}{2}}\right)$

$= -\csc\left(v^{\frac{1}{2}}\right)\cot\left(v^{\frac{1}{2}}\right)\cdot\frac{1}{2}v^{-\frac{1}{2}}$

$= -\frac{\csc(\sqrt{v})\cot(\sqrt{v})}{2\sqrt{v}}$

Question

Find $g'(t)$ where $g(t) = \sqrt[5]{t}$.

$g(t) = t^{\frac{1}{5}}$ $\quad g'(t) = \frac{1}{5} t^{-\frac{4}{5}}$

(a) $g'(t) = 5t^4$ left

(b) $g'(t) = -5t^{-6}$ right

(c) $g'(t) = \dfrac{1}{5t^{4/5}}$ both $= \dfrac{1}{5\sqrt[5]{t^4}}$

(d) $g'(t) = \dfrac{1}{5}t^{4/5}$ neither

Inverse Functions

Suppose $y = f(x)$ and $x = g(y)$ are inverse functions—i.e. $(g \circ f)(x) = g(f(x)) = x$ for all x in the domain of f.

Theorem: Let f be differentiable on an open interval containing the number x_0. If $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$. Moreover

$$\frac{d}{dy} g(y_0) = g'(y_0) = \frac{1}{f'(x_0)}.$$

$(x_0, f(x_0))$

Note that this refers to a pair (x_0, y_0) on the graph of f—i.e. (y_0, x_0) on the graph of g. The slope of the curve of f at this point is the reciprocal of the slope of the curve of g at the associated point.

Example

The function $f(x) = x^7 + x + 1$ has an inverse function g. Determine $g'(3)$.

y value

$$g'(3) = \frac{1}{f'(\_\_)}$$

x value that goes with y=3

For what x is $f(x)=3$?

$$x^7 + x + 1 = 3$$

$$x = 1 \qquad f(1) = 3$$

(1, 3) point on graph of f

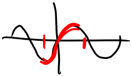
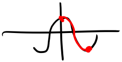
$$g'(3) = \frac{1}{f'(1)}$$

$$f'(x) = 7x^6 + 1$$

$f'(1) = 7(1)^6 + 1 = 8$

$g'(3) = \frac{1}{8}$

Inverse Trigonometric Functions

Recall the definitions of the inverse trigonometric functions.

angle

$$y = \sin^{-1} x \iff x = \sin y, \quad -1 \le x \le 1, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

ratio

$$y = \cos^{-1} x \iff x = \cos y, \quad -1 \le x \le 1, \quad 0 \le y \le \pi$$

$$y = \tan^{-1} x \iff x = \tan y, \quad -\infty < x < \infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Inverse Trigonometric Functions

There are different conventions used for the ranges of the remaining functions. Sullivan and Miranda use

$$y = \cot^{-1} x \quad \Longleftrightarrow \quad x = \cot y, \quad -\infty < x < \infty, \quad 0 < y < \pi$$

$$y = \csc^{-1} x \quad \Longleftrightarrow \quad x = \csc y, \quad |x| \geq 1, \quad y \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \Longleftrightarrow \quad x = \sec y, \quad |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Derivative of the Inverse Sine

Use implicit differentiation to find $\frac{d}{dx}\sin^{-1}x$, and determine the interval over which $y=\sin^{-1}x$ is differentiable.

$\boxed{y=\sin^{-1}x} \longleftrightarrow \sin y = x$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

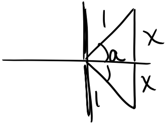
$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos \boxed{y}}$$

$$\frac{dy}{dx} = \frac{1}{\boxed{\cos(\sin^{-1}x)}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^{-1} x = \sin^{-1}\left(\frac{x}{1}\right)$$



$$a^2 + x^2 = 1^2$$

$$a^2 = 1 - x^2$$

$$a = \pm\sqrt{1-x^2}$$

must be positive

$$a = \sqrt{1-x^2}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\cos(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{1}$$

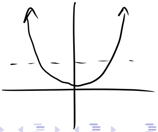
$$= \sqrt{1-x^2}$$

Where is $\sin^{-1} x$ differentiable?

· Rephrased: What is the domain of $y = \frac{1}{\sqrt{1-x^2}}$?

$$1 - x^2 > 0$$

$$x^2 < 1$$



$$-1 < x < 1$$

Examples

$$\frac{d}{du}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}$$

Evaluate each derivative

(a) $\displaystyle \frac{d}{dx}\sin^{-1}(e^x)$

outside: $\sin^{-1}(\quad)$

inside: e^x

$$= \frac{1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x$$

$$= \frac{e^x}{\sqrt{1-e^{2x}}}$$

Examples

Evaluate each derivative

(b) $\dfrac{d}{dx}\left(\sin^{-1} x\right)^3$

outside: $(\quad)^3$

inside: $\sin^{-1} x$

$$= 3\left(\sin^{-1} x\right)^2 \cdot \frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= 3\left(\sin^{-1} x\right)^2 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{3\left(\sin^{-1} x\right)^2}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sin^{-1}(e^x)\right)^3$$

$$= 3\left(\sin^{-1}(e^x)\right)^2 \frac{d}{dx}\left(\sin^{-1}(e^x)\right)$$

$$= 3\left(\sin^{-1}(e^x)\right)^2 \frac{1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x)$$

$$= 3\left(\sin^{-1}(e^x)\right)^2 \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$$

Derivative of the Inverse Tangent

Theorem: If $f(x) = \tan^{-1} x$, then f is differentiable for all real x and

$$f'(x) = \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.$$

The argument uses implicit differentiation just like we used for the inverse sine function. **It is left as an EXERCISE.**

Question: $\frac{d}{du} \tan^{-1} u = \frac{1}{1+u^2}$

Find $\frac{dy}{dx}$ where $y = \tan^{-1}(e^x)$.

$$\frac{dy}{dx} = \frac{1}{1+(e^x)^2} \cdot \frac{d}{dx}(e^x)$$

(a) $\dfrac{dy}{dx} = \dfrac{e^x}{1+e^{2x}}$ right

(b) $\dfrac{dy}{dx} = \dfrac{e^x}{1+x^2}$ left

(c) $\dfrac{dy}{dx} = \dfrac{1}{1+e^{2x}}$ both

(d) $\dfrac{dy}{dx} = \sec^{-2}(e^x)$ neither

Derivative of the Inverse Secant

Theorem: If $f(x) = \sec^{-1} x$, then f is differentiable for all $|x| > 1$ and

$$f'(x) = \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}.$$

Examples

Evaluate

(a) $\dfrac{d}{dx}\sec^{-1}(x^2) = \dfrac{1}{(x^2)\sqrt{(x^2)^2-1}} \cdot \dfrac{d}{dx}(x^2) = \dfrac{2\cancel{x}}{x^{\cancel{2}}\sqrt{x^4-1}}$

$$= \frac{2}{x\sqrt{x^4-1}}$$

(b) $\dfrac{d}{dx}\tan^{-1}(\sec x)$

The Remaining Inverse Functions

Due to the trigonometric cofunction identities, it can be shown that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

and

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$