February 28 Math 3260 sec. 51 Spring 2020

Section 4.1: Vector Spaces and Subspaces

Definition A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Remarks

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1. is that V is **closed** under (a.k.a. with respect to) vector addition.
- Property 6. is that V is **closed** under scalar multiplication.
- A vector space has the same basic *structure* as $\mathbb{R}^n$
- These are **axioms**. We assume (not "prove") that they hold for vector space V. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.

Examples of Vector Spaces

For an integer $n \geq 0$, $\mathbb{P}_n$ denotes the set of all polynomials with real coefficients of degree at most n. That is

in

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[1] and scalar multiplication are defined by

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

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[1] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

Example

What is the zero vector **0** in $\mathbb{P}_n$?

Let $\mathbf{0}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$. Find the values of $a_0, \ldots, a_n$.

We know that $(\vec{0} + \vec{p})(t) = \vec{p}(t)$

Let $\vec{p}(t) = p_0 + p_1 t + \ldots + p_n t^n$

$$(\vec{0} + \vec{p})(t) = \vec{0}(t) + \vec{p}(t) = (a_0 + p_0) + (a_1 + p_1)t + \ldots + (a_n + p_n)t^n$$

$$= p_0 + p_1 t + \ldots + p_n t^n$$

$\Rightarrow a_0 + p_0 = p_0 \quad a_0 = 0, \quad a_1 + p_1 = p_1 \quad a_1 = 0$

In fact, $a_i + p_i = p_i$ so $a_i = 0$ for all i

So $\vec{0}(t) = 0 + 0t + 0t^2 + \ldots + 0t^n = 0$

Example

If $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$, what is the vector $-\mathbf{p}$?

Let $-\mathbf{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$. Find the values of $c_0, \ldots, c_n$.

We know $\vec{p} + (-\vec{p}) = \vec{0}$

$$(\vec{p} + (-\vec{p}))(t) = \vec{p}(t) + (-\vec{p})(t)$$

$$= (p_0 + c_0) + (p_1 + c_1)t + \ldots + (p_n + c_n)t^n$$

$$= 0 + 0t + \ldots + 0t^n$$

$\Rightarrow p_0 + c_0 = 0 \Rightarrow c_0 = -p_0$, $p_1 + c_1 = 0 \Rightarrow c_1 = -p_1$

In general $c_i = -p_i$ for each $i = 0, \ldots, n$

$$-\vec{p}(t) = -p_0 - p_1 t - p_2 t^2 - \ldots - p_n t^n$$

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(1) Does property 1. hold for V?

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be in V

so $x \leq 0$, $y \leq 0$, $a \leq 0$ and $b \leq 0$.

$\vec{u} + \vec{v} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$ Is this in V?

$x + a \leq 0 + 0 = 0$ and $y + b \leq 0 + 0 = 0$

Yes, $\vec{u} + \vec{v}$ is in V. V is closed under vector addition.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(2) Does property 6. hold for V?

Consider $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and scaler c

$c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$. Is this in V?

Not necessarily. If $c < 0$ and at least one of x, y is non zero, then $c\vec{u}$ isn't in V.

e.g. $\vec{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $c = -2$ $\quad c\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

V is not closed under scalar multiplication.

Theorem

Let V be a vector space. For each $\mathbf{u}$ in V and scalar c

$$
\begin{aligned}
0\mathbf{u} &= \mathbf{0} \\
c\mathbf{0} &= \mathbf{0} \\
-1\mathbf{u} &= -\mathbf{u}
\end{aligned}
$$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in H[2]

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[2] This is sometimes replaced with the condition that H is nonempty.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

Does it satisfy the three properties?

① Is $\vec{0}$ in the set?

$\vec{0} = (0,0)$ which has the form $(u_1, 0)$

where $u_1 = 0$.

② Is it closed under vector addition?

Let $\vec{u} = (u_1, 0)$ and $\vec{v} = (v_1, 0)$

$\vec{u} + \vec{v} = (u_1 + v_1, 0+0) = (u_1 + v_1, 0)$

$\vec{u}+\vec{v}$ has zero as its second component.

The set is closed under vector addition.

(3) Is it closed under scalar multiplication?

Let c be any scalar

$$c\vec{u} = (cu_1, c\cdot 0) = (cu_1, 0)$$

Yes, this is in the set too. The set is closed under scalar multiplication.

All three properties hold, the set is a subspace of $\mathbb{R}^2$.

Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

Does it satisfy the properties?

① Is $\vec{0}$ in the set?

Is $\vec{0} = (0,0)$ of the form $(u_1, 1)$?

$(0,0) \neq (u_1, 1)$ for any

choice of u_1 since $0 \neq 1$.

It doesn't contain the zero vector, it is not a subspace of $\mathbb{R}^2$.