February 2 Math 1190 sec. 63 Spring 2017

Section 1.5: Infinite Limits, Limits at Infinity, Asymptotes

Here, we will consider limits involving the symbols ∞ (*infinity*) and $-\infty$ (*negative infinity*).

They will be used to denote **unboundedness** in the positive and negative directions, respectively.

Infinities: Arithmetic and Indeterminate Forms

While ∞ and $-\infty$ are NOT NUMBERS, there is an arithmetic for using these symbols. In particular

- $\infty + \infty = \infty$
- $\infty + c = \infty$ for any real number c
- $\infty \cdot c = \infty$ if $c > 0$ and $\infty \cdot c = -\infty$ if $c < 0$
- $\frac{0}{\infty} = \frac{0}{-\infty} = 0$

Other forms that may appear are indeterminate. The following **are not defined.**

$$\infty - \infty, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty^0$$

Infinite Limits

Investigate the limit

$$\lim_{x\to 0} \frac{1}{x^2}$$

x	$f(x) = \frac{1}{x^2}$
-0.1	100
-0.01	10000
-0.001	1000000
0	undefined
0.001	1000000
0.01	10000
0.1	100

grow without bound as $x \to 0$

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Infinite Limits

Definition: Let $f(x)$ be defined on an open interval containing c except possibly at c. Then

$$\lim_{x \to c} f(x) = \infty$$

provided $f(x)$ can be made arbitrarily large by taking x sufficiently close to c. (The definition of

$$\lim_{x \to c} f(x) = -\infty$$

is similar except that f can be made arbitrarily large and negative.)

The top limit statement reads *the limit as x approaches c of $f(x)$ equals infinity*.

Infinite Limits

Investigate the limit

$$\lim_{x \to 0} \frac{1}{x}$$

$\lim_{x \to 0} \frac{1}{x}$ DNE

x	$f(x) = \frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
0	undefined
0.001	1000
0.01	100
0.1	10

growing without bound but negative

growing without bound but positive

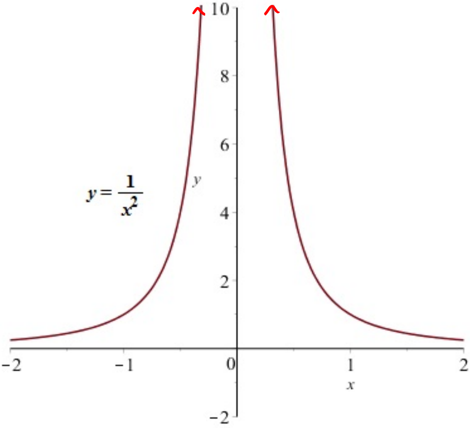
we can say

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

and

$\lim_{x \to 0^+} \frac{1}{x} = \infty$

Infinite Limits and Graphs



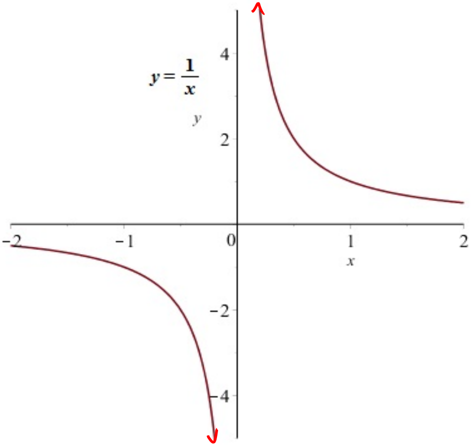


Figure: If f has an infinite limit at a finite number c, then the graph has a vertical asymptote.

An Observation

Suppose when taking a limit $\lim_{x \to c} f(x)$ we see the form

$$\frac{k}{0} \quad \text{where } k \text{ is any nonzero real number.}$$

Then this limit **MAY** be either ∞ or $-\infty$.

- If we determine that the ratio is positive for all x near c, the limit is ∞.
- If we determine that the ratio is negative for all x near c, the limit is $-\infty$.
- If the ratio can take either sign for x sufficiently close to c, the limit DNE.

Evaluate Each Limit if Possible

(a) $\lim_{x\to 1^-} \frac{2x+1}{x-1}$

as $x \to 1^-$, $2x+1 \to 3$
which is positive.
$x-1$ is going to zero.

$x \to 1^-$ means x is less than 1. $x<1 \Rightarrow x-1<0$
the denominator goes to zero through
negative numbers.
The form is "$\frac{3}{0}$" with $\frac{+}{-}$ which is negative.

So $\lim_{x\to 1^-} \frac{2x+1}{x-1} = -\infty$

(b) $\lim_{x \to 1^+} \dfrac{2x+1}{x-1}$

$2x+1 \to 3$ which is still positive.

The denominator is still headed to zero, so the form is "$\frac{3}{0}$"

$x \to 1^+$ means x is greater than 1. $x > 1 \Rightarrow x - 1 > 0$ positive

The ratio looks like $\frac{+}{+}$ which is positive.

Hence

$$\lim_{x \to 1^+} \frac{2x+1}{x-1} = \infty$$

(c) $\lim_{x \to 1} \frac{2x+1}{x-1}$ DNE

The numerator heads to 3 which is positive.

The denominator is going to zero, but the sign could be + or −.

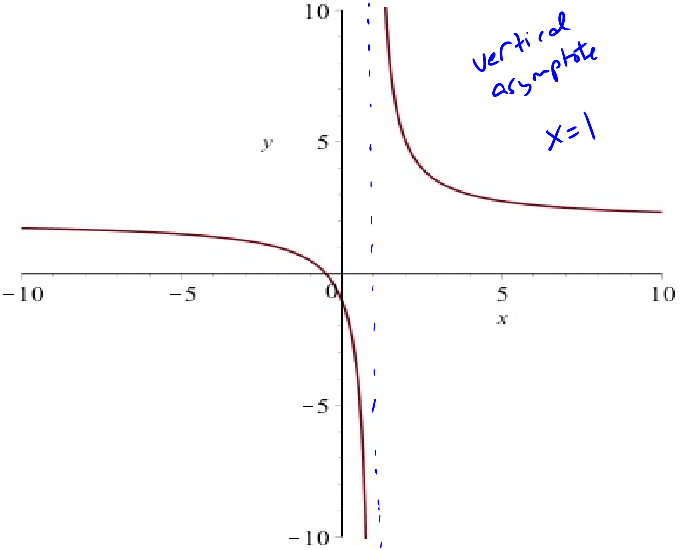


Figure: $f(x) = \frac{2x+1}{x-1}$

Question

Evaluate if possible $\lim_{t\to 2} \frac{3}{(t-2)^2}$.

(a) ∞

(b) $-\infty$

(c) 4

(d) DNE

Note 3 is positive, and due to the square $(t-2)^2$ is positive.

The form is "$\frac{3}{0}$" with $\frac{+}{+}$ which is positive.

Well Known Infinite Limits

Some limits that follow from what we know about these functions

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

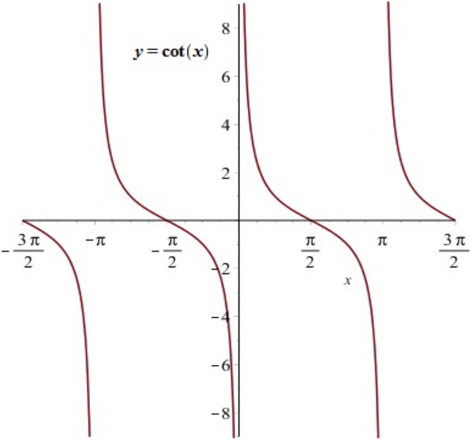
$$\lim_{\theta\to\frac{\pi}{2}^-} \tan\theta = \infty \quad \text{and} \quad \lim_{\theta\to\frac{\pi}{2}^+} \tan\theta = -\infty$$

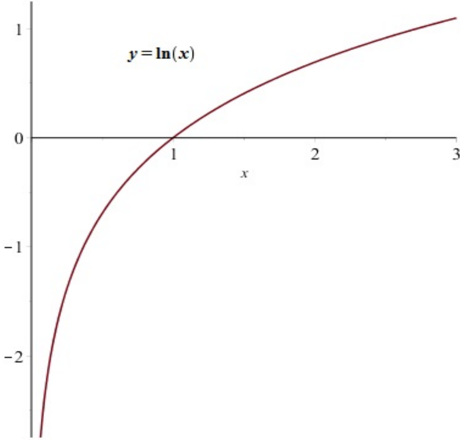
$$\lim_{\theta\to 0^-} \cot\theta = -\infty \quad \text{and} \quad \lim_{\theta\to 0^+} \cot\theta = \infty$$

$$\lim_{\theta\to\frac{\pi}{2}^-} \sec\theta = \infty \quad \text{and} \quad \lim_{\theta\to\frac{\pi}{2}^+} \sec\theta = -\infty$$

$$\lim_{\theta\to 0^-} \csc\theta = -\infty \quad \text{and} \quad \lim_{\theta\to 0^+} \csc\theta = \infty$$

Infinite Limits and Graphs





Limits at Infinity

We know what is meant by a limit being infinite (i.e. $f \to \infty$ or $f \to -\infty$). Now, we want to consider limits like

$$\lim_{x\to\infty} f(x) \qquad \text{or like}$$

$$\lim_{x\to-\infty} f(x).$$

What is meant by such a thing, and how is it related to a function's graph?

Definitions

Let f be defined on an interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large.

Similarly

Defintion: Let f be defined on an interval $(-\infty, a)$. Then

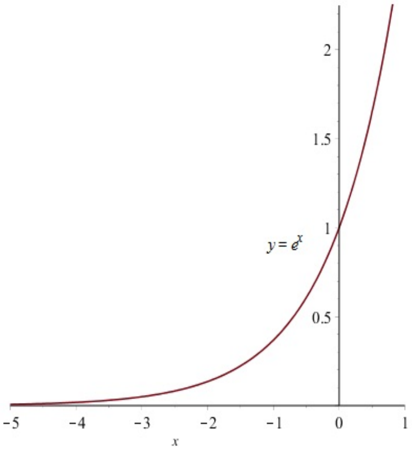
$$\lim_{x\to-\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large and negative.

Example

Investigate the limit

$$\lim_{x \to -\infty} e^x$$



At the far left, the graph hugs the x-axis, the line $y=0$.

$$\lim_{x \to -\infty} e^x = 0$$

Some Results to Remember

Let k be any real number and let p be rational. Then

$$\lim_{x\to\infty} \frac{k}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{k}{x^p} = 0$$

The latter holds assuming x^p is defined for $x < 0$.

e.g. $\lim_{x\to\infty} \frac{7}{x^4} = 0$ or $\lim_{x\to-\infty} \frac{\pi}{x^6} = 0$

In essence, if the numerator of a ratio is staying finite while the denominator is becoming infinite, the ratio is tending to zero.

Examples

Evaluate if possible

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2+5x+2}$$

We know $\lim_{x\to\infty} \frac{k}{x^p} = 0$ for p-positive

We identify the largest power of x in the denominator.

Here it's x^2. We multiply by $\frac{1}{x^2} / \frac{1}{x^2}$.

$$\lim_{x\to\infty} \left(\frac{3x^2+2x-1}{x^2+5x+2}\right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{2}{x^2}}$$

Distribute

$$= \lim_{x\to\infty} \frac{3 + \frac{2}{x} - \frac{1}{x^2}}{1 + \frac{5}{x} + \frac{2}{x^2}}$$

Cancel as needed

$$= \frac{3+0-0}{1+0+0} = \frac{3}{1} = 3$$

* $\lim_{x\to\infty} x = \infty$ $\quad \lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to\infty} x^p = \infty$

for p positive.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$$

Again, we'll find the highest power of x in the denominator and multiply and divide by its reciprocal.

$$= \lim_{x\to\infty} \left(\frac{\sqrt{x^2+1}}{x+1}\right) \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{x^2+1}}{1+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}}\sqrt{x^2+1}}{1+\frac{1}{x}} \quad \text{by } ①$$

Note:

① If $x > 0$, then

$$x = \sqrt{x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(x^2+1)}}{1+\frac{1}{x}}$$

② $\sqrt{a}\sqrt{b} = \sqrt{a \cdot b}$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \frac{\sqrt{1+0}}{1+0} = \frac{\sqrt{1}}{1} = 1$$

* Note if $x < 0$ then $x = -\sqrt{x^2}$ *

Question

Evaluate if possible

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+2x}}{4x+3}$$

$$= \lim_{x\to\infty} \left(\frac{\sqrt{3x^2+2x}}{4x+3}\right) \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{3x^2+2x}}{4+3/x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(3x^2+2x)}}{4+\frac{3}{x}} = \lim_{x\to\infty} \frac{\sqrt{3+2/x}}{4+\frac{3}{x}} = \frac{\sqrt{3+0}}{4+0} = \frac{\sqrt{3}}{4}$$

(a) DNE

(b) $\frac{3}{4}$

(c) $\sqrt{3}$

(d) $\frac{\sqrt{3}}{4}$

Infinte Limits at Infinity

The following limits may arise

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Two critical limits to remember (YOU'LL NEED TO KNOW THESE)

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to\infty} \ln(x) = \infty$$

also $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$

Vertical and Horizontal Asymptotes

Vertical Asymptotes: The line $x = c$ is a *vertical asymptote* to the graph of f if

$$\lim_{x\to c^+} f(x) = \pm\infty, \quad \text{or} \quad \lim_{x\to c^-} f(x) = \pm\infty.$$

Horizontal Asymptotes: The line $y = L$ is a *horizontal asymptote* to the graph of f if

$$\lim_{x\to\infty} f(x) = L, \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

A good candidate for a vertical asymptote would be a number that makes a denominator zero.

Questions

(1) **True or False:** Since $\lim_{x \to 0^+} \ln(x) = -\infty$, we can conclude that the line $x = 0$ is a vertical asymptote to the graph of $y = \ln(x)$.

(2) **True or False:** Since $\lim_{x \to -\infty} e^x = 0$, we can conclude that the line $y = 0$ is a horizontal asymptote to the graph of $y = e^x$.