# Feb. 2 Math 2254H sec 015H Spring 2015

## Section 7.3 Trigonometric Substitution

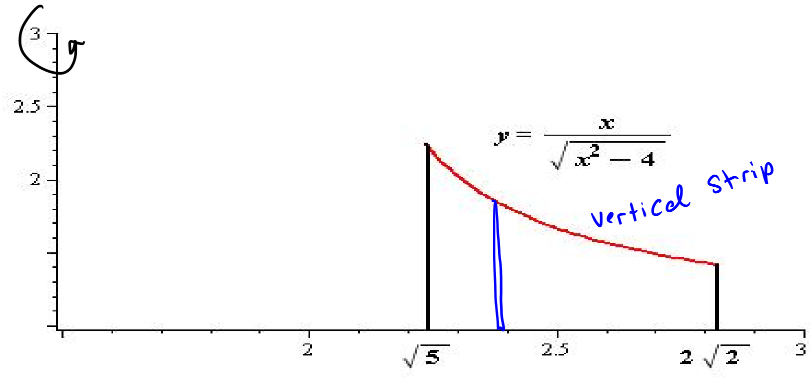
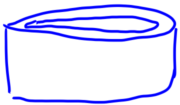



Figure: The region bounded below $y = x/\sqrt{x^2-4}$ for $\sqrt{5} \le x \le \sqrt{8}$ is rotated about the $y$-axis to form a solid.

# Find the volume of the solid.



One shell w/ thickness $\Delta x$

$$V_{shell} = 2\pi r h \Delta x$$

height $h = \frac{x}{\sqrt{x^2-4}}$, radius $r = x$

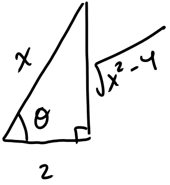
Total Volume $V = \int_{\sqrt{5}}^{\sqrt{8}} 2\pi \frac{x^2}{\sqrt{x^2-4}}\, dx$

$$\int \frac{x^2}{\sqrt{x^2-4}}\,dx$$

$$= \int \frac{(2\sec\theta)^2\,\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{2}\tan\cancel{\theta}}$$

$$= 4\int \sec^3\theta\,d\theta$$

Int by parts




$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-4} = 2\tan\theta$

$$= 4\sec\theta\tan\theta - 4\int \sec\theta\tan^2\theta\,d\theta$$

$u = \sec\theta \qquad du = \sec\theta\tan\theta\,d\theta$

$v = \tan\theta \qquad dv = \sec^2\theta\,d\theta$

$$= 4\sec\theta\tan\theta - 4\int \sec\theta\,(\sec^2\theta - 1)\,d\theta$$

$$= 4\sec\theta\tan\theta - 4\int \sec^3\theta\,d\theta + 4\int\sec\theta\,d\theta \implies$$

$$8\int \sec^3\theta\,d\theta = 4\sec\theta\tan\theta + 4\ln|\sec\theta + \tan\theta| + C$$

$$\Rightarrow 4\int \sec^3\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| + K$$

$$\int \frac{x^2}{\sqrt{x^2-4}}\,dx = 2\left(\frac{x}{2}\right)\frac{\sqrt{x^2-4}}{2} + 2\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + K$$

$$2\pi\int_{\sqrt5}^{\sqrt8} \frac{x^2}{\sqrt{x^2-4}}\,dx = \pi x\sqrt{x^2-4}\,\Big|_{\sqrt5}^{\sqrt8} + 4\pi\ln\left|x + \sqrt{x^2-4}\right|\,\Big|_{\sqrt5}^{\sqrt8}$$

$$= 4\sqrt2\,\pi - \sqrt5\,\pi + 4\pi\ln(\sqrt8 + 2) - 4\pi\ln(\sqrt5 + 1)$$

# Section 7.4: Rational Functions, Partial Fractions

Simplify

$$\frac{1}{x-3} - \frac{2}{x+4} \qquad = \frac{x+4}{(x-3)(x+4)} - \frac{2(x-3)}{(x-3)(x+4)}$$

$$= \frac{-x+10}{x^2+x-12}$$

Now evaluate the integral

$$\int \frac{10-x}{x^2+x-12}\,dx \quad = \int \left( \frac{1}{x-3} - \frac{2}{x+4} \right) dx$$

$$= \ln|x-3| - 2\ln|x+4| + C$$

# We sort'a cheated! The big question is:

If we started with the simplified *total fraction*

$$\frac{-x+10}{x^2+x-12},$$

how could we figure out that it *decomposes* into the sum of the smaller *partial fractions*

$$\frac{1}{x-3} - \frac{2}{x+4}?$$

# Rational Functions

Recall that a rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P$ and $Q$ are polynomials.

The function is called a **proper rational function** if

$$\text{degree}(P(x)) \; < \; \text{degree}(Q(x)).$$

# Rational Functions

If degree($P(x)$) $\geq$ degree($Q(x)$), then $f$ is an **improper rational function.** In this case, we can write

$$f(x) = p(x) + \frac{r(x)}{Q(x)}$$

where $p$ is a polynomial, and $r(x)/Q(x)$ is proper. We can obtain this using long division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

# Decomposing Proper Rational Functions

**Theorem:** Every polynomial $Q(x)$ with real coefficients can be factored into a product

$$Q(x) = q_1(x)q_2(x)\cdots q_k(x)$$

where each $q_i$ is either a linear factor (i.e. $q_i(x) = ax + b$) or an irreducible quadratic (i.e. $q_i(x) = ax^2 + bx + c$ where $b^2 - 4ac < 0$).

Knowing that such a factorization exists, and being able to compute it are two different animals! But at least we can know that the cases to be outlined cover all contingencies.

# Decomposing Proper Rational Functions

Let $f(x) = P(x)/Q(x)$ be a **proper** rational function, and let $Q(x)$ be factored completely into linear and irreducible quadratic factors

$$f(x) = \frac{P(x)}{q_1(x)q_2(x)\cdots q_k(x)}.$$

We'll consider four cases

(i) each factor of $Q$ is linear and none are repeated,

(ii) each factor of $Q$ is linear and one or more is repeated,

(iii) some factor(s) of $Q$ are quadratic, but no quadratic is repeated,

(iv) $Q$ has at least one repeated quadratic factor.

# Case (i) Non-repeated Linear Factors

Suppose $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. And no pair of $a$'s and $b$'s (both) match. Then we look for a decomposition of $f$ in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

For example

$$\frac{10 - x}{(x - 3)(x + 4)} = \frac{A}{x - 3} + \frac{B}{x + 4}.$$

Note that each fraction in the expansion is a proper rational function with denominator a line.

# Example: Evaluate the integral

$$\int \frac{4x-2}{x^3-x}\,dx$$

Partial Fractions:

$$\frac{4x-2}{x^3-x} = \frac{4x-2}{x(x^2-1)} = \frac{4x-2}{x(x-1)(x+1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad \cdot(x^3-x)$$

Clear fractions:

$$4x-2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$= A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$$\underbrace{0}x^2 + \underbrace{4}x \underbrace{-2} = \underbrace{(A+B+C)}x^2 + \underbrace{(B-C)}x \underbrace{-A}$$

$$A = 2, \qquad B - C = 4, \qquad A + B + C = 0$$

$$B + C = -2$$

$$2B = 2 \Rightarrow B = 1 \qquad 1 - C = 4 \Rightarrow C = -3$$

$$\int \frac{4x-2}{x^3-x}\,dx = \int \left( \frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1} \right) dx$$

$$= 2\ln|x| + \ln|x-1| - 3\ln|x+1| + C$$

$$4x - 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Set $x = 0$

$$-2 = A(-1)(1) \Rightarrow A = 2$$

$x = 1$

$$4 - 2 = B(1)(2) \Rightarrow B = 1$$

$x = -1$

$$-4 - 2 = C(-1)(-2) \Rightarrow C = -3$$

# Case (ii) A Repeated Linear Factor

Suppose $Q(x)$ has only linear factors, but that one of them is repeated. That is, suppose $(a_i x + b_i)^n$ is a factor of $Q$. Then for this term, the decomposition of $f$ will contain the $n$ terms

$$\frac{A_{i1}}{a_i x + b_i} + \frac{A_{i2}}{(a_i x + b_i)^2} + \cdots + \frac{A_{in}}{(a_i x + b_i)^n}.$$

For example,

$$\frac{3x^2 + 2x - 1}{(x+1)^2(x-2)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}.$$