February 2 Math 2306 sec 58 Spring 2016

Section 4: First Order Equations: Linear

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

Example

Solve the initial value problem.

$$\cos t\frac{dy}{dt} + \sin t\, y = 1, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \quad y(0) = -1$$

For $-\frac{\pi}{2} < t < \pi/2$, $\cos t > 0$.

Standard form: $\quad \frac{dy}{dt} + \frac{\sin t}{\cos t}\, y = \frac{1}{\cos t}$

$$\frac{dy}{dt} + \tan t\, y = \sec t \qquad P(t) = \tan t$$

$$\int P(t)\,dt = \int \tan t\, dt = \ln(\sec t)$$

$$\mu = e^{\int P(t)\,dt} = e^{\ln(\sec t)} = \sec t$$

$$\sec t \frac{dy}{dt} + \sec t \tan t \, y = \sec t \sec t$$

$$\frac{d}{dt}\left[\sec t \, y\right] = \sec^2 t$$

$$\int \frac{d}{dt}\left[\sec t \, y\right] dt = \int \sec^2 t \, dt$$

$$\sec t \, y = \tan t + C$$

$$y = \frac{\tan t + C}{\sec t} = \sin t + C \cos t$$

Now apply $y(0) = -1$

$$y(0) = \sin 0 + C \cos 0 = -1$$

$$C = -1$$

The solution to the IVP is

$$y = \sin t - \cos t \quad .$$

Section 5: First Order Equations Models and Applications

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We let $P(t)$ denote the population (density) at time t where t is in years with $t = 0$ corresponding to 2011. The statement that the population's growth rate is proportional to the population translated into

$$\frac{dP}{dt} = kP \quad \text{for some constant k.}$$

We also note that we are given

$$P(0) = 58 \quad \text{and} \quad P(1) = 89.$$

$$\frac{dP}{dt} = kP \quad , \qquad P(0) = 58 \qquad \text{and} \qquad P(1) = 89$$

Using separation of variables

$$\frac{1}{P}\frac{dP}{dt} = k \quad \Longrightarrow \quad \frac{1}{P}\,dP = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt \quad \Longrightarrow \quad \ln P = kt + C$$

(note $P > 0$).

Exponentiate $\qquad e^{\ln P} = e^{kt+C} = e^{C} e^{kt}$

So $P = Ae^{kt}$ where $A = e^{C}$

From $P(0) = 58$, $P(0) = Ae^{0} = 58 \Rightarrow A = 58$

$P(t) = 58e^{kt}$

From $P(1) = 89$ $P(1) = 58e^{k} = 89$

$$e^{k} = \frac{89}{58} \Rightarrow \ln e^{k} = \ln\frac{89}{58}$$

$$k = \ln\frac{89}{58}$$

Hence $P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}$

with t in years since 2011, 2021 corresponds to $t = 10$.

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4198.06$$

The population is about 4198 rabbits in 2021.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

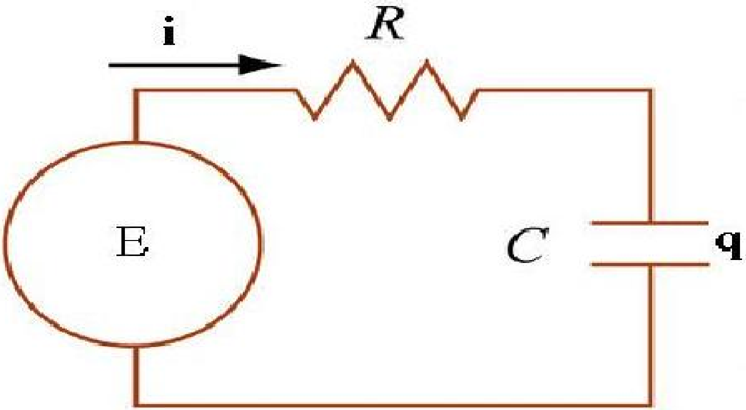

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

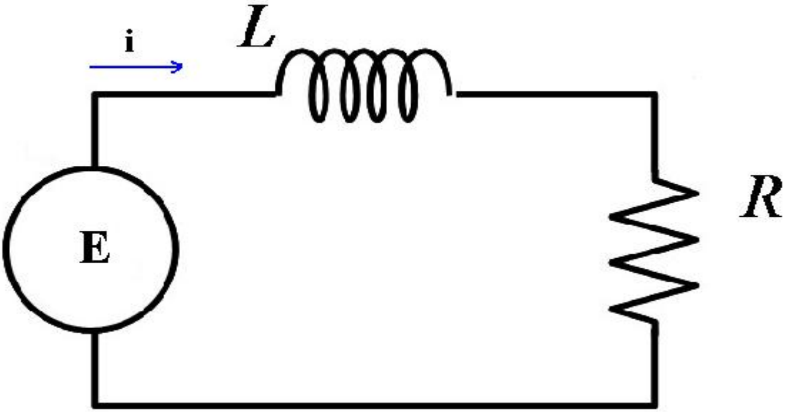

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),
Inductance L in henries (h),
Capacitance C in farads (f),
Implied voltage E in volts (V),
Charge q in coulombs (C),
Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

For an RC circuit

$$E = Ri + \frac{1}{C} q \qquad \text{as} \quad Ri = R\frac{dq}{dt}$$

we have

$$R\frac{dq}{dt} + \frac{1}{C} q = E$$

1st order linear

LR circuit

$$E = L\frac{di}{dt} + Ri$$

$$L\frac{di}{dt} + Ri = E$$

1st order

linear

(for current i)

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

Here $E(t) = 200$

$R = 1000$, $C = 5 \cdot 10^{-6}$

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200$$

$i(0) = q'(0) = 0.4$

Standard form $$\frac{dq}{dt} + \frac{10^6}{5 \cdot 1000}q = \frac{200}{1000}$$

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \quad q'(0) = 0.4$$

$$P(t) = 200 \qquad \int p(t)\,dt = \int 200\,dt = 200t$$

$$\mu = e^{\int p(t)\,dt} = e^{200t}$$

$$\frac{d}{dt}\left[e^{200t} q\right] = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left[e^{200t} q\right] dt = \int \frac{1}{5} e^{200t}\,dt$$

$$e^{200t} q = \frac{1}{5} \cdot \frac{1}{200} e^{200t} + K$$

$$q(t) = \frac{1}{1000} + k e^{-200t}$$

$$q'(t) = -200 k e^{-200t}, \quad q'(0) = 0.4$$

$$q'(0) = -200k e^{0} = 0.4 \Rightarrow k = \frac{0.4}{-200} = -\frac{\frac{2}{5}}{200} = \frac{-1}{500}$$

$$\boxed{\text{The charge } q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}.}$$

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500} e^{-200t}\right) = \frac{1}{1000} - 0 = \frac{1}{1000}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

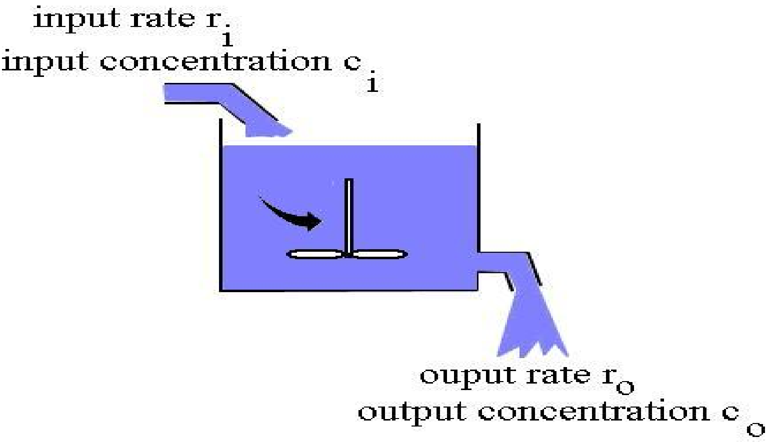

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array} \right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array} \right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

initial volume of the mixture

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

1st order linear equation.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

$V(0) = 500$ gal

$r_i = 5$ gal/min

$c_i = 2$ lb/gal

$r_o = 5$ gal/min

we'll take t in minutes

$$V(t) = V(0) + (r_i - r_o)t$$
$$= 500 \text{ gal}$$

From the pure water statement, $A(0) = 0$

Our IVP is $\dfrac{dA}{dt} + \dfrac{5\,\frac{gal}{min}}{500\,gal} A = 5\,\frac{gal}{min} \cdot 2\,\frac{lb}{gal}, \quad A(0)=0$

$$\frac{dA}{dt} + \frac{1}{100} A = 10, \quad A(0) = 0 .$$

$$P(t) = \frac{1}{100}, \qquad \mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] = 10\, e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 10 \cdot 100\, e^{\frac{1}{100}t} + C$$

$$A = 1000 + C e^{-\frac{1}{100}t}$$

From $A(0) = 0$, $\quad A(0) = 1000 + C e^{0} = 0 \Rightarrow C = -1000$

The amount of salt @ time t is

$$\boxed{A(t) = 1000 - 1000 e^{-\frac{1}{100}t}}$$

At $t=5$ min. the concentration in the tank is

$$\frac{A(5)}{V(5)} = \frac{1000-1000e^{-\frac{1}{100}\cdot 5}}{500}\frac{\text{lb}}{\text{gal}} \approx 0.098\ \frac{\text{lb}}{\text{gal}}$$