February 2 Math 2306 sec 59 Spring 2016

Section 4: First Order Equations: Linear

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

Example

Solve the initial value problem.

$$\cos t \frac{dy}{dt} + \sin t \, y = 1, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \quad y(0) = -1$$

For $-\pi/2 < t < \frac{\pi}{2}$, $\cos t > 0$

Standard form: $\dfrac{dy}{dt} + \dfrac{\sin t}{\cos t} y = \dfrac{1}{\cos t}$

$$\frac{dy}{dt} + \tan t \, y = \sec t$$

$P(t) = \tan t$, $\int P(t)\,dt = \int \tan t \, dt = \ln(\sec t)$

$$\mu(t) = e^{\int P(t)\,dt} = e^{\ln(\sec t)} = \sec t$$

$$\sec t \frac{dy}{dt} + \sec t \tan t\, y = \sec t \sec t$$

$$\frac{d}{dt}\left[\sec t\, y\right] = \sec^2 t$$

$$\int \frac{d}{dt}\left[\sec t\, y\right] dt = \int \sec^2 t\, dt$$

$$\sec t\, y = \tan t + C$$

$$y = \frac{\tan t + C}{\sec t} = \sin t + C \cos t$$

Apply $y(0) = -1$: $y(0) = \sin 0 + C \cos 0 = -1$

$C = -1$

The solution to the IVP is

$$\boxed{y = \sin t - \cos t.}$$

Section 5: First Order Equations Models and Applications

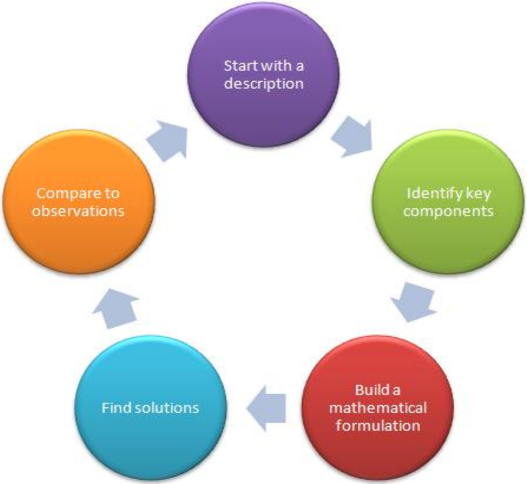

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let $P(t)$ be the number of rabbits at time t.

The rate at which P changes is $\frac{dP}{dt}$.

By the 1^{st} sentence, $\frac{dP}{dt} \propto P$

i.e. $\frac{dP}{dt} = kP$ for some constant k.

Let's take t in years and take $t=0$ in 2011.

Then $P(0)=58$ and $P(1)=89$.

Let's solve the IVP $\frac{dP}{dt}=kP$, $P(0)=58$

Separating the variables

$$\frac{1}{P}\frac{dP}{dt}=k \Rightarrow \frac{1}{P}\,dP=k\,dt$$

$$\int\frac{1}{P}\,dP=\int k\,dt \Rightarrow \ln P=kt+C$$ (note $P>0$)

$$e^{\ln P}=e^{kt+C} \qquad P=e^{C}e^{kt}=Ae^{kt}$$ where $A=e^{C}$

From $P(0)=58$, $\quad P(0)=Ae^0=58 \Rightarrow A=58$

Hence $\quad p(t)=58e^{kt}$.

From $P(1)=89$, $\quad P(1)=58e^k=89$

$$e^k=\frac{89}{58} \Rightarrow \ln e^k=\ln\frac{89}{58} \Rightarrow k=\ln\frac{89}{58}$$

The population at time t is

$$p(t)=58e^{t\ln\frac{89}{58}}$$

Since 2021 corresponds to $t=10$, the estimated 2021 population is

$$P(10) = 58\, e^{10 \ln \frac{89}{58}} \approx 4198$$

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

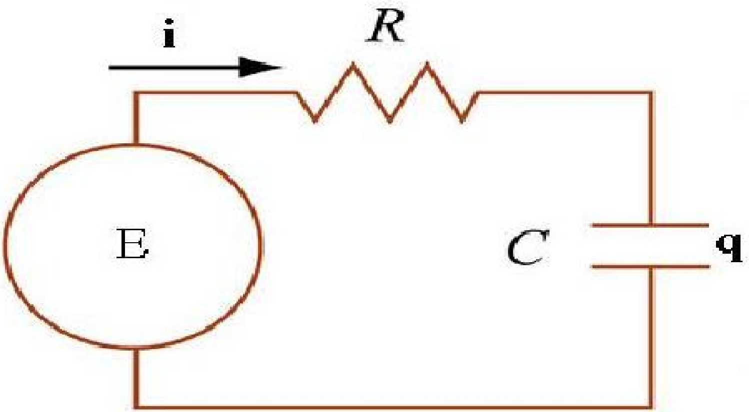

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

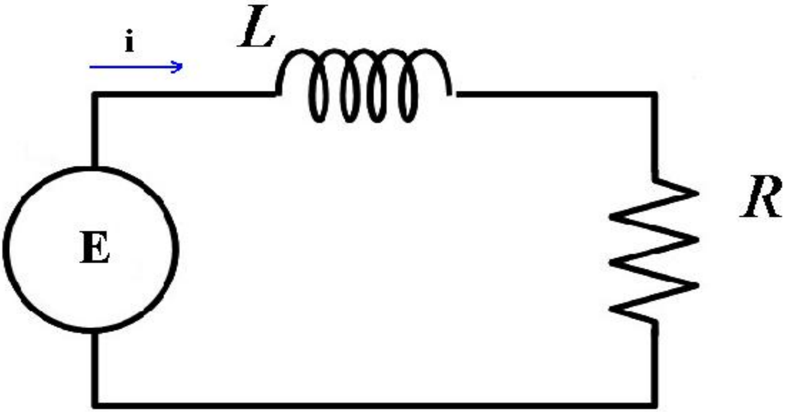

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Inductance L in henries (h), Capacitance C in farads (f), Implied voltage E in volts (V), Charge q in coulombs (C), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

For an RC-circuit

$$E = Ri + \frac{1}{C}q = R\frac{dq}{dt} + \frac{1}{C}q$$

i.e. $$R\frac{dq}{dt} + \frac{1}{C}q = E$$

1st order linear for $q(t)$.

LR-circuit

$$E = L\frac{di}{dt} + Ri$$

$$L\frac{di}{dt} + Ri = E$$

1st order linear for $i(t)$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

$E = 200$ V

$R = 1000\,\Omega$

$C = 5 \cdot 10^{-6}$ f

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200$$

$i(0) = q'(0) = 0.4$ A

$$\frac{dq}{dt} + \frac{10^6}{5 \cdot 10^3}q = \frac{200}{1000}$$

$q'(0) = 0.4$

$$\frac{dq}{dt} + 200q = \frac{1}{5} \qquad q'(0) = 0.4$$

$$P(t) = 200, \qquad \mu = e^{\int P(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$e^{200t} q' + 200 e^{200t} q = \frac{1}{5} e^{200t}$$

$$\frac{d}{dt}\left[e^{200t} q\right] = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left[e^{200t} q\right] dt = \int \frac{1}{5} e^{200t}\,dt = \frac{1}{5} \cdot \frac{1}{200} e^{200t} + K$$

$$e^{200t} q = \frac{1}{1000} e^{200t} + k$$

$$q = \frac{1}{1000} + k e^{-200t}, \quad q'(t) = -200 k e^{-200t}$$

$$q'(0) = 0.4 \Rightarrow q'(0) = -200 k e^{0} = 0.4 \Rightarrow k = \frac{0.4}{-200} = -\frac{1}{500}.$$

So the charge at time t is

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}.}$$

"C" for Coulombs

Long time charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \left(\frac{1}{1000} - \frac{1}{500} e^{-200t} \right) = \frac{1}{1000} - 0 = \frac{1}{1000} \quad C$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

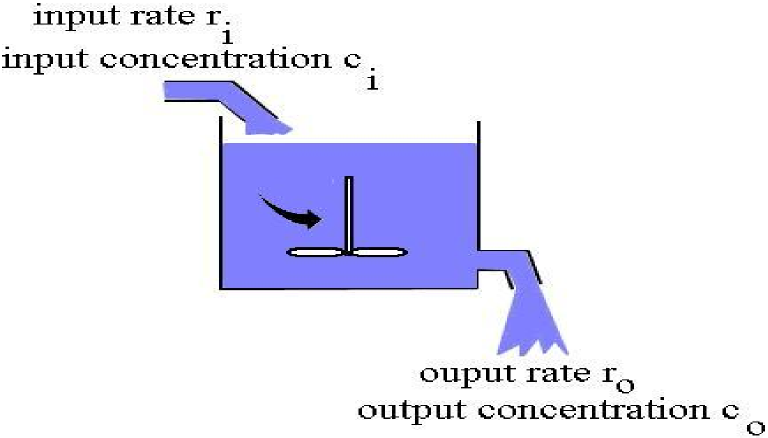

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array}\right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

initial volume

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

1st order linear for A.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$$V(0) = 500 \text{ gal}, \quad r_i = 5 \frac{\text{gal}}{\text{min}}, \quad r_o = 5 \frac{\text{gal}}{\text{min}}$$

$$c_i = 2 \frac{\text{lb}}{\text{gal}}, \quad c_o = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t} = \frac{A(t)}{500} \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

$$\frac{dA}{dt} + \frac{5\,\frac{\text{gal}}{\text{min}}}{500\ \text{gal}} A\ \text{lb} = 5\,\frac{\text{gal}}{\text{min}} \cdot 2\,\frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} + \frac{1}{100} A = 10, \qquad A(0) = 0$$

originally, the tank contains pure water.

$$P(t) = \frac{1}{100}, \quad \mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] = 10\,e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] dt = \int 10\,e^{\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}t} A = 10 \cdot 100\, e^{\frac{1}{100}t} + C$$

$$A(t) = 1000 + C e^{-\frac{1}{100}t}$$

From $A(0)=0$, $\quad A(0) = 1000 + Ce^{0} = 0$

$$C = -1000$$

The amount of salt $A(t)$ in the tank is

$$\boxed{A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}.}$$

The concentration in the tank at $t = 5$ min.

is

$$\frac{A(5)}{V(5)} = \frac{1000 - 1000\, e^{-\frac{1}{100}\cdot 5}}{500} \ \frac{\text{lb}}{\text{gal}}$$

$$= 0.098$$