February 2 Math 2335 sec 51 Spring 2016

Section 3.1: Root Finding, Bisection Method

Many problems in the sciences, business, manufacturing, etc. can be framed in the form:

Given a function $f(x)$, find an input value c such that

$$f(c) = 0.$$

For example:

- $f(x) = F'(x)$ and we are looking for a maximum or minimum value of F, or
- $f(x) = h(x) - g(x)$ and we are looking for a solution of the equation $g(x) = h(x)$, or
- f is a polynomial with large degree, and we are looking for its roots

Motivating Example

In the absence of immigration, the rate of change of a certain population is jointly proportional to the current population and the difference between the population level and the maximum number that the environment can support. The population increases due to immigration, but the rate of increase diminishes exponentially as the number of individuals increases.

Letting $P(t)$ be the population at time t, a model of this process is

$$\frac{dP}{dt} = P(1-P) + e^{-P}.$$

Question: Is there an equilibrium population level? (i.e. one for which $P'(t) = 0$.)

Root Finding Problems

Answering the question amounts to solving the equation

$$P(1-P)+e^{-P}=0,$$

an equation for which there is no convenient algebraic approach.

Definition: The number $x = c$ is called a **zero** of the function $f(x)$ provided $f(c) = 0$. We will also call c a **root** of the equation $f(x) = 0$.

Bisection Method

We will consider several iterative methods for root finding. To say that a method is *iterative* means we will

- Construct a sequence of steps intended to find a solution,
- Include an *exit strategy* step (test to see if the problem is solved)
- Repeat the steps until the exit criterion is met

The first method (subject of section 3.1) is called the **Bisection Method**. We will assume throughout that the function $f(x)$ is continuous on the interval $a \leq x \leq b$, and that $f(a)f(b) < 0$.

Bisection Method Example:

Problem: Find a positive number c that is a zero of the function

$$f(x) = x(1-x) + e^{-x}.$$

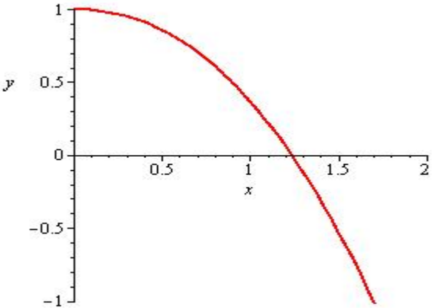


Figure: Plot of $f(x) = x(1-x) + e^{-x}$ for $0 \le x \le 2$

Intermediate Value Theorem

Theorem: Suppose f is continuous on $[a, b]$, and let L be any number between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) such that $f(c) = L$.

In particular, if $f(a)$ and $f(b)$ have opposite signs, and f is continuous on $a \leq x \leq b$, then it's graph must have an x-intercept. This means that 0 is a number between $f(a)$ and $f(b)$ so that the IVT guarantees at least one root of the equation $f(x) = 0$ on the interval.

How Bisection Works

i. We start by finding a and b such that $f(a)$ and $f(b)$ have opposite signs. And we set an error tolerance $\epsilon > 0$.

ii. We set $c = (a + b)/2$, the midpoint of the interval.

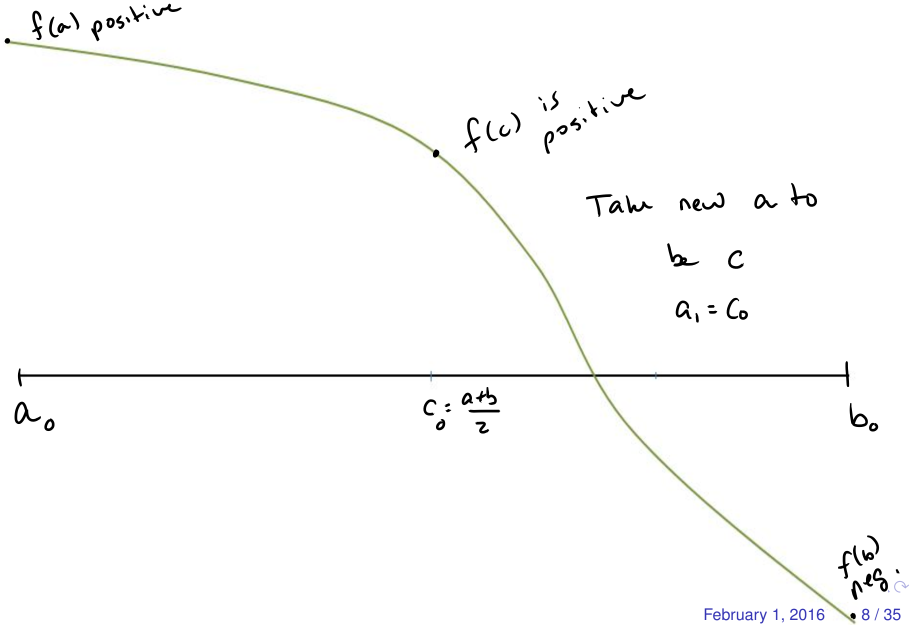
iii. If $b - c < \epsilon$, we accept c as the root and stop[1].

iv. If $f(b)f(c) \leq 0$, set $a = c$. Otherwise, set $b = c$. Return to step ii.

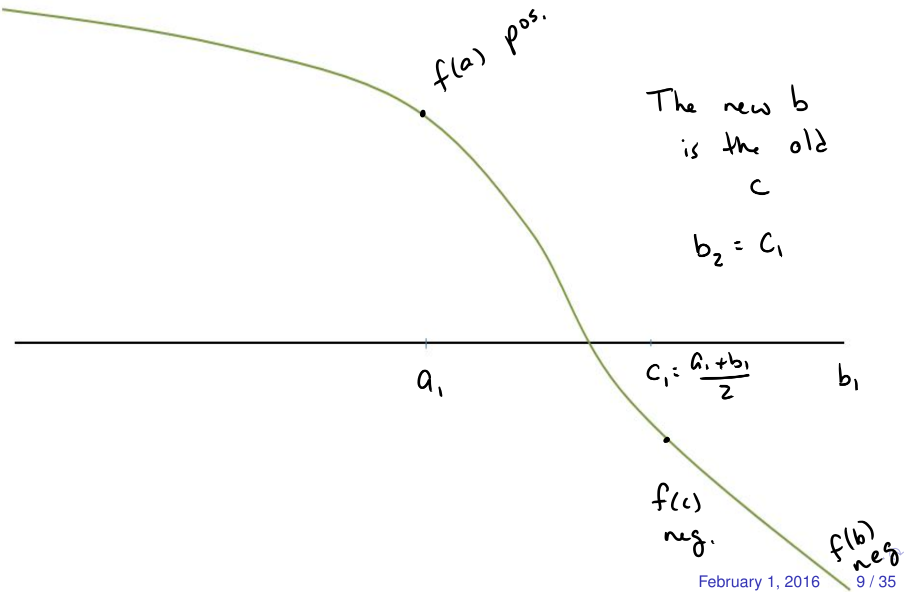
Note that after each iteration, our interval is half the length from the previous step. So we are guaranteed that our interval length will eventually be less than ϵ, and we will produce the root.

[1] This is the exit strategy.

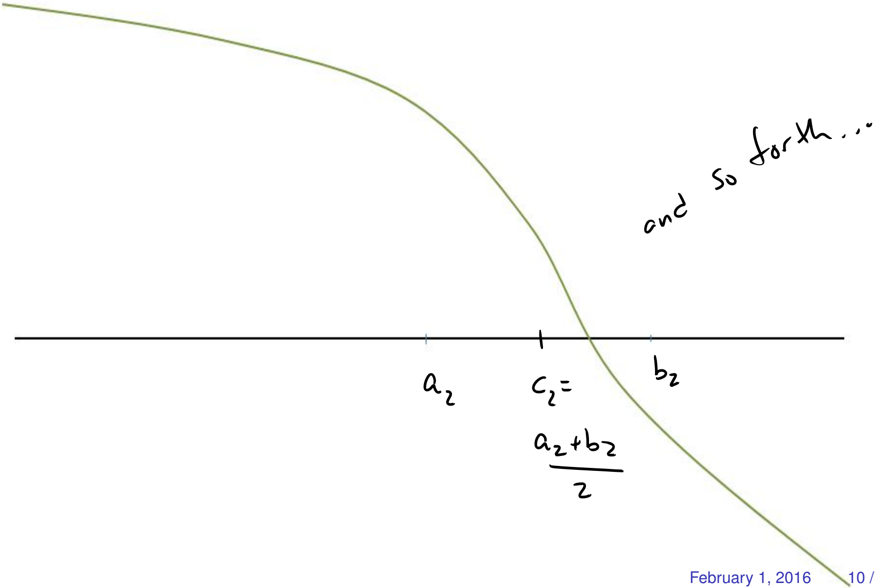
How Bisection Works (original iterate–zeroth)



How Bisection Works (next iterate–first)



How Bisection Works (second iterate)



Bisection Method Example

Solve $x(1-x)+e^{-x}=0$ using $[a,b]=[0,2]$ with an error tolerance of $\epsilon=0.1$.

$f(x) = e^{-x} + x(1-x)$, $f(0) = 1$, $f(2) \doteq -1.8647$

$a_0 = 0$ $\quad c_0 = \frac{2+0}{2} = 1$ $\quad f(1) \doteq 0.3679$

$b_0 = 2$

$f(c_0)$ has the same sign as $f(a_0)$

so $a_1 = c_0 = 1$

Note $b_0 - c_0 = 1 > 0.1$

$a_1 = 1 \qquad c_1 = \frac{1+2}{2} = \frac{3}{2} \qquad f(a_1) = 0.3679$

$b_1 = 2 \qquad\qquad\qquad\qquad f(b_1) = -1.8647$

$$f(c_1) = f\left(\tfrac{3}{2}\right) \doteq -0.5269$$

So $\quad b_2 = c_1 = 3/2$

$$b_1 - c_1 = 2 - \tfrac{3}{2} = 0.5 > 0.1$$

$a_2 = 1$ $\qquad f(a_2) = 0.3679$

$b_2 = \frac{3}{2}$ $\qquad f(b_2) = -0.5269$

$c_2 = \frac{3/2+1}{2} = \frac{5}{4}$ $\qquad f(c_2) = f\left(\frac{5}{4}\right) \doteq -0.0260$

so $\quad b_3 = c_2 = \frac{5}{4}$

$$b_2 - c_2 = \frac{3}{2} - \frac{5}{4} = \frac{1}{4} = 0.25 > 0.1$$

$a_3 = 1$ $\quad f(a_3) = 0.3679$

$b_3 = \frac{5}{4}$ $\quad f(b_3) = -0.0260$

$c_3 = \frac{1+\frac{5}{4}}{2} = \frac{9}{8}$ $\quad f(c_3) = 0.1840$

$$a_4 = c_3 = \frac{9}{8}$$

$$b_3 - c_3 = \frac{5}{4} - \frac{9}{8} = \frac{1}{8} = 0.125 > 0.1$$

$$a_4 = \frac{9}{8} \qquad f(a_4) = 0.1840$$

$$b_4 = \frac{5}{4} \qquad f(b_4) = -0.0260$$

$$c_4 = \frac{9/8 + \frac{5}{4}}{2} = \frac{19}{16} \qquad f(c_4) = 0.0823$$

$$a_5 = c_4 = \frac{19}{16}$$

$$b_4 - c_4 = \frac{5}{4} - \frac{19}{16} = \frac{1}{16} = 0.0625 < 0.1$$

The error tolerance is met. We take the root to be $c_4 = \frac{19}{16} = 1.1875$

$x(1-x)+e^{-x}=0$ Bisection Method in Matlab ®

Take $a=0$, $b=2$, and set $\epsilon=0.001$.

a	b	c	$b-c$	$f(c)$
0	2.0000	1.0000	1.0000	0.3679
1.0000	2.0000	1.5000	0.5000	−0.5269
1.0000	1.5000	1.2500	0.2500	−0.0260
1.0000	1.2500	1.1250	0.1250	0.1840
1.1250	1.2500	1.1875	0.0625	0.0823
1.1875	1.2500	1.2188	0.0313	0.0290
1.2188	1.2500	1.2344	0.0156	0.0017
1.2344	1.2500	1.2422	0.0078	−0.0121
1.2344	1.2422	1.2383	0.0039	−0.0052
1.2344	1.2383	1.2363	0.0020	−0.0017
1.2344	1.2363	1.2354	0.0010	−0.0000

The true root to seven decimal places is $\alpha=1.2353462$.

Analysis of the Bisection Method

Strength: If there is one root in the interval $[a, b]$, the bisection method will always find it to within the set tolerance!

Weakness: The method is slow. It takes many more iterations than some other methods for a given tolerance.

We can determine the number of iterations required as it depends on a, b and ϵ.

Error Bound for the Bisection Method

Let α denote the exact value of the solution. And let a_n, b_n and c_n be the computed values of a, b, and c at the n^{th} iteration.

We seek a bound on the error $|\alpha - c_n|$ at the n^{th} step.

Note

$$|\alpha - c_n| < |b_n - a_n|$$

Note

$$b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$$

$$b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2}\cdot\frac{1}{2}(b_0 - a_0) = \frac{1}{2^2}(b_0 - a_0)$$

$$b_3 - a_3 = \frac{1}{2}(b_2 - a_2) = \frac{1}{2}\cdot\frac{1}{2^2}(b_0 - a_0) = \frac{1}{2^3}(b_0 - a_0)$$

$$\vdots$$

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$$

From $|\alpha - c_n| < |b_n - a_n|$ we get the bound

$$|\alpha - c_n| < \frac{1}{2^n}(b_0 - a_0)$$

Number of Iterations Needed: Bisection Method

From $|\alpha - c_n| \le \frac{1}{2^n}(b-a)$, show that $|\alpha - c_n| \le \epsilon$ provided

$$n \ge \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2}.$$

Since $|\alpha - c_n| < \frac{1}{2^n}(b-a)$, if $\frac{1}{2^n}(b-a) < \varepsilon$

we are guaranteed to have

$$|\alpha - c_n| < \varepsilon .$$

From $\quad \frac{1}{2^n}(b-a) < \varepsilon$

$$\frac{b-a}{\varepsilon} < 2^n$$

$$\ln 2^n > \ln\left(\frac{b-a}{\varepsilon}\right)$$

$$n \ln 2 > \ln\left(\frac{b-a}{\varepsilon}\right) \implies$$

$$n > \frac{\ln\left(\frac{b-a}{\varepsilon}\right)}{\ln 2}.$$

Number of Iterations Needed: Bisection Method

For a given tolerance ϵ, we determined that the number of iterations n must satisfy

$$n \geq \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2}.$$

Example: Find an interval $[a, b]$ containing the smallest positive solution of $\cos(x) = \frac{1}{2} + \sin(x)$. Determine the minimum number of iterations needed to find the solution by the bisection method to within a tolerance of $\epsilon = 10^{-5}$.

Let $f(x) = \cos x - \frac{1}{2} - \sin x$.

Then a root of f solves the equation

$$\cos x = \frac{1}{2} + \sin x.$$

$$f(0) = \cos 0 - \tfrac{1}{2} - \sin 0 = 1 - \tfrac{1}{2} = \tfrac{1}{2} > 0.$$

$$f\left(\tfrac{\pi}{2}\right) = \cos\tfrac{\pi}{2} - \tfrac{1}{2} - \sin\tfrac{\pi}{2} = -\tfrac{1}{2} - 1 = \tfrac{-3}{2} < 0$$

A root must be in $\left[0, \frac{\pi}{2}\right]$. For this choice of $[a,b]$ we have

$$n > \frac{\ln\left(\frac{\frac{\pi}{2} - 0}{10^{-5}}\right)}{\ln 2} = \frac{\ln\left(\frac{\pi}{2}\cdot 10^{5}\right)}{\ln 2} \doteq 17.26$$

Taking $n=18$ iterates will give the desired accuracy.

Example Continued...

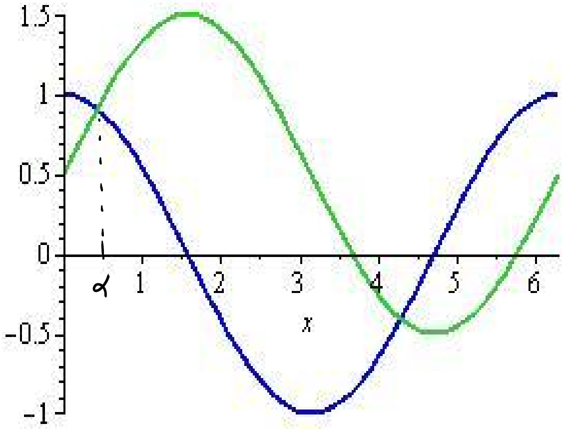


Figure: Plot of $h(x) = \cos(x)$ (blue) and $g(x) = \frac{1}{2} + \sin(x)$ (green).

Example

Find an interval $[a, b]$ containing the real solution of

$$x^3 = x^2 + x + 1$$

and determine the minimum number of iterations of the bisection method required to find the solution within a tolerance of $\epsilon = 10^{-8}$.

Let $f(x) = x^3 - x^2 - x - 1$

$f(0) = -1$, $f(-1) = -1 - 1 + 1 - 1 = -2$, $f(-2) = -8 - 4 + 2 - 1 = -11$

$f(1) = 1 - 1 - 1 - 1 = -2$, $f(2) = 8 - 4 - 2 - 1 = 1$

We get a sign change for f on $[1, 2]$.

This choice for $[a,b]$ will require

$$n > \frac{\ln\left(\frac{2-1}{10^{-8}}\right)}{\ln 2} = \frac{\ln(10^8)}{\ln 2} \doteq 26.58$$

Taking 27 iterates will give the desired accuracy.