February 3 MATH 1112 sec. 54 Spring 2020

Exponential Functions

Definition: Let a be a positive real number different from 1—i.e. $a > 0$ and $a \neq 1$. The function

$$f(x) = a^x$$

is called the **exponential function of base** a. Its domain is $(-\infty, \infty)$, and its range is $(0, \infty)$.

Some examples of exponential functions are

$$f(x) = 2^x, \qquad g(x) = \left(\frac{1}{3}\right)^x, \quad \text{and} \quad h(x) = \pi^{x-1}$$

The Logarithm Function of Base a

Definition: Let $a > 0$ and $a \neq 1$. For $x > 0$ define $\log_a(x)$ as a number such that

$$\text{if} \quad y = \log_a(x) \quad \text{then} \quad x = a^y.$$

The function

$$F(x) = \log_a(x)$$

is called the **logarithm function of base** a. It has domain $(0, \infty)$, range $(-\infty, \infty)$, and if $f(x) = a^x$ then

$$F(x) = f^{-1}(x).$$

Composition of Exponentials and Logarithms

Let $a > 0$ and $a \neq 1$

For every real number x

$$\log_a(a^x) = x$$

For every $x > 0$

$$a^{\log_a(x)} = x$$

Recall that for a one to one function f with inverse f^{-1}

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x.$$

Graph of Logarithms

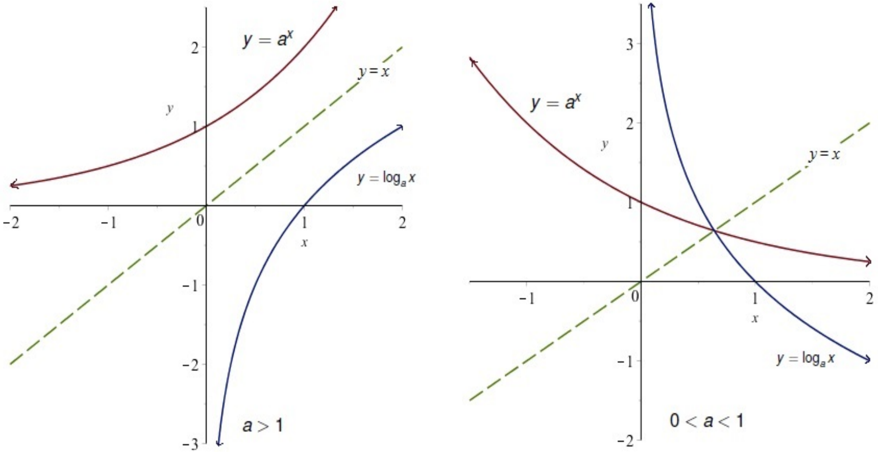

Figure: The graph of a logarithm can be obtained from the graph of an exponential by reflection in the line $y = x$. There are two cases depending on whether $0 < a < 1$ or $a > 1$.

Question

Recall: $y = \log_a(x)$ means $x = a^y$.

If $y = \log_3\left(\frac{1}{9}\right)$, then

(a) $y = \frac{1}{2}$

(b) $y = -\frac{1}{2}$

(c) $y = -2$

(d) y is undefined.

Since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Evaluating Logs with a Calculator

Most logarithms can't be evaluated by hand. For example, $\log_e(70) \approx 4.2485$. No one memorizes that! Calculators tend to have two built in logarithm functions.

- **Common Log** base 10 denoted as $\log$ (note there is no subscript), and
- **Natural Log** base e denoted[1] ln

In Calculus, you'll find that the prefered base is e—the natural log. Base 10 logs still have some use such as in the Richter scale used to measure earthquakes.

[1] The order LN instead of NL is probably due to the French name *le Logarithme Naturel* for this log.

Evaluating Logs with a Calculator

Theorem: (Change of Base) Let a, b, and M be any positive numbers, then

$$\log_b(M) = \frac{\log_a(M)}{\log_a(b)}.$$

What this says is that you can turn a $\log_2$ problem into a $\log$ or $\ln$ problem, and use your machine!

Example:

$$\log_2(15) = \frac{\ln(15)}{\ln(2)} \approx 3.9069 \quad \text{(courtesy of TI-84).}$$

Question

Here's the meat of our theorem again: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$.

Suppose you wish to evaluate $\log_\pi(42)$ using a calculator. Which of the following expressions could be used to find the desired value?

(a) $\dfrac{\log(42)}{\log(\pi)}$

$$\log_\pi(42) = \frac{\log(42)}{\log(\pi)}$$

(b) $\ln\left(\dfrac{42}{\pi}\right)$

(c) $\dfrac{\ln(\pi)}{\ln(42)}$

(d) $\ln\left(\dfrac{\pi}{42}\right)$

Properties of Exponentials and Logarithms

Let $a > 0$, with $a \neq 1$. Then for any real x and y

- Exponential of a Sum: $a^{x+y} = a^x \cdot a^y$
- Exponential of a Difference: $a^{x-y} = \dfrac{a^x}{a^y}$
- Power of an Exponential: $(a^x)^y = a^{xy}$

Properties of Exponentials and Logarithms

Let $a > 0$, with $a \neq 1$. Then for any real, positive numbers M and N and for any rational number p

- Log of Product: $\log_a(MN) = \log_a(M) + \log_a(N)$
- Log of Quotient: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$
- Log of Power: $\log_a(M^p) = p\log_a(M)$

Question

Here's the meat of our theorem: $\log_a(MN) = \log_a(M) + \log_a(N)$.

Which of the following is equivalent to $\log_3(15)$?

(a) $\log_3(5) + \log_3(3)$

$15 = 5 \cdot 3$

$\log_3(5 \cdot 3) = \log_3(5) + \log_3(3)$

(b) $\log_3(10) + \log_3(5)$

(c) $\log_3(3) \cdot \log_3(5)$

(d) all of the above are equivalent

(e) none of the above is equivalent

Question

Here's the meat of our theorem: $\log_a(M^p) = p\log_a(M)$.

Which of the following is equivalent to $\log_7(125)$?

(a) $5\log_7(25)$

(b) $3\log_7(5)$

(c) $(\log_7(5))^3$

(d) all of the above are equivalent

(e) none of the above is equivalent

$125 = 5^3$

$\log_7(5^3) = 3\log_7(5)$

Question

Here's the meat of our theorem: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$.

Which of the following is equivalent to $\log_3(5)$?

(a) $\log_3(15) - \log_3(3)$

$5 = \frac{15}{3}$

(b) $\log_3(10) - \log_3(5)$

$\log_3(5) = \log_3\left(\frac{15}{3}\right)$

(c) $\frac{\log_3(15)}{\log_3(3)}$

$= \log_3(15) - \log_3(3)$

(d) all of the above are equivalent

(e) none of the above is equivalent

Question

Properties of logarithms are sometimes confused with erroneous, somewhat look-alike expressions.
Which of the following is FALSE?

(a) $\ln(xy) = (\ln x)(\ln y)$

(b) $\log_2(x) = \ln(x^2)$

(c) $\log_4(2+7) = \log_4(2) + \log_4(7)$

(d) $(\log(10))^5 = \log(10^5)$

(e) All of the above are false.

Assume each expression is well defined.

(i) Change of base: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$

(ii) Log of Product: $\log_a(MN) = \log_a(M) + \log_a(N)$

(iii) Log of Power: $\log_a(M^p) = p\log_a(M)$

(iv) Log of Quotient: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$

(v) Inverse Function: $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$

(vi) Special Values: $\log_a(1) = 0$, $\log_a(a) = 1$, and $\log_a(0)$ is never defined.