February 3 Math 3260 sec. 51 Spring 2020

Section 1.9: The Matrix for a Linear Transformation

Elementary Vectors: We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in $\mathbb{R}^3$ they would be

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity I_n.

Matrix of Linear Transformation

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ be a linear transformation, and suppose

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_2\mathbf{e}_1 + x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

since T is a linear Transformation

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Letting $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix}$, then A

is a 4×2 matrix such that

$$T(\vec{x}) = A\vec{x}$$

for any $\vec{x}$ in $\mathbb{R}^2$.

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)].$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling trasformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

Here, we're in $\mathbb{R}^2$ so $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$T(\vec{e}_1) = r\vec{e}_1 = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \quad \text{and}$$

$$T(\vec{e}_2) = r\vec{e}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

So the standard matrix

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

Let's confirm:

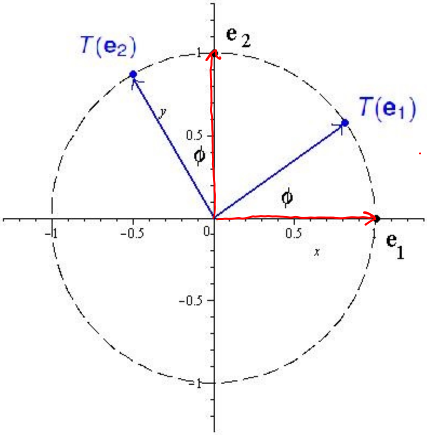
Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 + 0x_2 \\ 0x_1 + rx_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$

$$= r\vec{x}$$

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ. Find the standard matrix for T.



Using some basic trigonometry, the points on the unit circle

$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

$$= (-\sin\phi, \cos\phi)$$

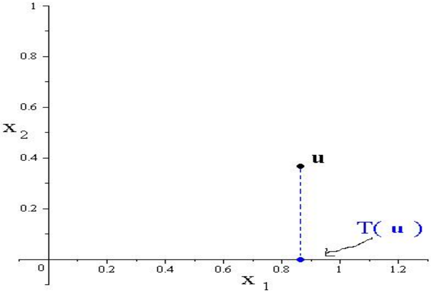
So $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.

Example[1]

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection transformation that projects each point onto the x_1 axis

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Find the standard matrix for T.



[1] See pages 73–75 in Lay for matrices associated with other geometric tranformation on $\mathbb{R}^2$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The Property **Onto**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b}$$

is always solvable. If T is a linear transformation with standard matrix A, then this is equivalent to saying $A\mathbf{x} = \mathbf{b}$ is always consistent.

Determine if the transformation is onto.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

call this A

A is 2×3, so $T : \mathbb{R}^3 \to \mathbb{R}^2$

Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^2$?

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. An augment matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{array}\right]$$

not a pivot column for any choice of b_1, b_2

Yes, $A\vec{x} = \vec{b}$ is always consistent.

That is $\vec{b}$ is in the range of T

for every $\vec{b}$ in $\mathbb{R}^2$.

T is <u>onto</u>.

The Property **One to One**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$

Determine if the transformation is one to one.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

Does $T(\vec{x}) = T(\vec{y})$ imply that $\vec{x} = \vec{y}$?

Suppose $T(\vec{x}) = T(\vec{y})$. Since T is linear

$$T(\vec{x}) = T(\vec{y}) \Rightarrow \quad T(\vec{x}) - T(\vec{y}) = \vec{0}$$

$$\Rightarrow \quad T(\vec{x} - \vec{y}) = \vec{0}$$

Consider the homogeneous equation $T(\vec{z}) = \vec{0}$

(note $\vec{z} = \vec{x} - \vec{y}$)

An augmented matrix for $A\vec{z}=\vec{0}$ is

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

We have two pivot columns and one non-pivot.

$z_1 = -2z_3$
$z_2 = -3z_3$
z_3 - free

Any $\vec{z} = z_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ is a solution

$\vec{x}-\vec{y}$ need not be zero!

We've found infinitely many solutions to $T(\vec{z})=\vec{0}$. Hence T is <u>not</u> one to one.

Some Theorems on Onto and One to One

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

→ for onto check $A\vec{x} = \vec{b}$ for consistency

for one to one check $A\vec{x} = \vec{0}$ for nontrivial soln.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

$T : \mathbb{R}^2 \to \mathbb{R}^3$.

$T(\vec{e}_1) = T(1,0) = (1,2,0)$ so the matrix

$T(\vec{e}_2) = T(0,1) = (0,-1,3)$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

Check to see if T is one to one.

Consider $A\vec{x} = \vec{0}$ (no free variables - one to one)

Using an augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are no free variables $A\vec{x} = \vec{0}$ if and only if $\vec{x} = \vec{0}$

The mapping <u>is</u> one to one.

Is T onto?

Is $A\vec{x} = \vec{b}$ always consistent?

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & 6b_1 - 3b_2 - b_3 \end{bmatrix}$$

The 3^{rd} column is a pivot column except when $6b_1 - 3b_2 - b_3 = 0$.

So $A\vec{x} = \vec{b}$ is only consistent when $b_1 = \frac{1}{2} b_2 + \frac{1}{6} b_3$.

Since $T(\vec{x}) = \vec{b}$ is not

always consistent, T is <u>not</u> onto.

To recap, $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_2 \end{bmatrix}$$

has standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

It is one to one, but

it is not onto.