February 3 Math 3260 sec. 55 Spring 2020

Section 1.9: The Matrix for a Linear Transformation

Elementary Vectors: We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in $\mathbb{R}^3$ they would be

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity I_n.

Matrix of Linear Transformation

Let $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^4$ be a linear transformation, and suppose

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_2\mathbf{e}_1 + x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2)$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2)$$

Since T is linear

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

given how the matrix-vector product is defined

So $T(\vec{x}) = A\vec{x}$ if $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^4$

makes A a 4×2 matrix

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]\,.$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling trasformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T.

We need $\vec{e}_1$ and $\vec{e}_2$ from $\mathbb{R}^2$ and $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $T(\vec{e}_1) = r\,\vec{e}_1 = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $T(\vec{e}_2) = r\,\vec{e}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$

So the standard matrix $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$.

Lets verify that this is correct:

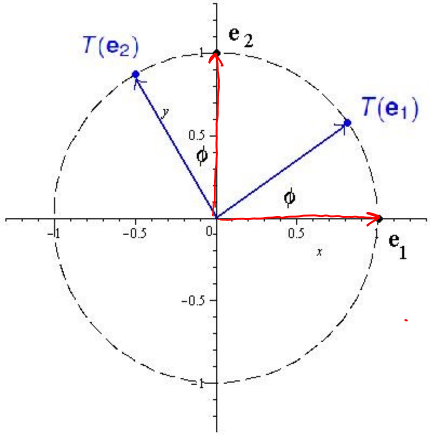
$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 + 0 \\ 0 + rx_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix} = r\vec{x}$

$= T(\vec{x})$

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ. Find the standard matrix for T.



Using some basic trigonometry, the points on the unit circle

$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

$$= (-\sin\phi, \cos\phi)$$

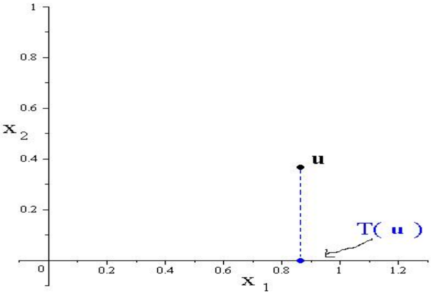
So $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$

Example[1]

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection transformation that projects each point onto the x_1 axis

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Find the standard matrix for T.



[1]See pages 73–75 in Lay for matrices associated with other geometric tranformation on $\mathbb{R}^2$

We need $T(\vec{e}_1)$ and $T(\vec{e}_2)$

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The standard matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The Property **Onto**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b}$$

is always solvable. If T is a linear transformation with standard matrix A, then this is equivalent to saying $A\mathbf{x} = \mathbf{b}$ is always consistent.

Determine if the transformation is onto.

$$T(\mathbf{x}) = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}}_{A} \mathbf{x}.$$

Note A is 2×3 so $T: \mathbb{R}^3 \to \mathbb{R}^2$

Is $T(\vec{x}) = \vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^2$?

i.e. is $A\vec{x} = \vec{b}$ consistent for all $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$?

We can use an augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

the last column is never a pivot column

$A\vec{x}=\vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^2$.

So T is onto.

The Property **One to One**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$

Determine if the transformation is one to one.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

Does $T(\vec{x}) = T(\vec{y})$ require $\vec{x} = \vec{y}$?

Suppose $T(\vec{x}) = T(\vec{y})$. Since T is linear

$$T(\vec{x}) = T(\vec{y}) \Rightarrow T(\vec{x}) - T(\vec{y}) = \vec{0}$$

$$T(\vec{x} - \vec{y}) = \vec{0}$$

Let $\vec{z} = \vec{x} - \vec{y}$ and consider the homogeneous equation $T(\vec{z}) = \vec{0}$ — i.e. $A\vec{z} = \vec{0}$

we have the augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

we'd have a free variable

$z_1 = -2z_3$

$z_2 = -3z_3$

z_3 - free

$T(\vec{z}) = \vec{0}$ has nontrivial solutions so

$T(\vec{x})$ can equal $T(\vec{y})$ even if $\vec{x} \neq \vec{y}$.

So T is not one to one.

Some Theorems on Onto and One to One

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_2 \end{bmatrix}, \quad T: \mathbb{R}^2 \to \mathbb{R}^3$$

Let's find the standard matrix A for T.

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

To show T is one to one, we can show that $A\vec{x} = \vec{0}$ has only the trivial solution.

The augmented matrix $[A \ \vec{0}]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ has no free variables

$A\vec{x} = \vec{0}$ has only the trivial solution, hence T is one to one.

We'll come back and finish this next time.