February 4 Math 2306 sec. 54 Spring 2019

Section 4: Exact Equations

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

Suppose that the equation $M\,dx + N\,dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x,y) = xy^2$.

$$xy^2(2y-6x)\,dx + xy^2(3x-4x^2y^{-1})\,dy = 0, \quad \Longrightarrow$$

$$(2xy^3 - 6x^2y^2)\,dx + (3x^2y^2 - 4x^3y)\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial(\mu M)}{\partial y} &= 6xy^2 - 12x^2y \\ \frac{\partial(\mu N)}{\partial x} &= 6xy^2 - 12x^2y \end{aligned}\right\} \text{equal}$$

This new equation is exact

Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers n and m). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on x, or there is an integrating factor $\mu = \mu(y)$ depending only on y.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M\,dx + N\,dy = 0$$

is NOT exact, but that

$$\mu M\,dx + \mu N\,dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on y. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

Let's use the product rule in the right side.

Special Integrating Factor $\mu = \mu(x)$

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y}$$ product rule

$$= 0 \cdot M + \mu \frac{\partial M}{\partial y}$$ $\mu = \mu(x)$ so $\frac{\partial}{\partial y}\mu = 0$

$$= \mu \frac{\partial M}{\partial y}$$

$$\frac{\partial(\mu N)}{\partial x} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$ product rule

$$= \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x}$$ $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$ since $\mu = \mu(x)$

$$\frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x} = \mu \frac{\partial M}{\partial y}$$

$$\frac{d\mu}{dx} N = \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x}$$

$$= \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \mu$$

For $N(x,y) \neq 0$

$$\frac{d\mu}{dx} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \mu$$

Separable ODE for μ

This ODE is solvable for $\mu(x)$ only if the coefficient depends <u>only</u> on x.

In this case, we solve for μ and get

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\, dx \right)$$

Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{1}$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is an special integrating factor for (1). If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is an special integrating factor for (1).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

Check for exactness: $M(x,y) = 2xy$ and $N(x,y) = y^2 - 3x^2$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = -6x \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Not exact

We look for an integrating factor $\mu(x)$ or $\mu(y)$

$$x: \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - (-6x)}{y^2 - 3x^2} = \frac{8x}{y^2 - 3x^2}$$

Depends on y

There is no IF a function of x.

$$y: \quad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x - 2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y}$$

This is a function of only y.

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\, dy\right) = e^{\int \frac{-4}{y} dy}$$

$$= e^{-4\ln|y|} = e^{\ln y^{-4}} = y^{-4}$$

Multiply ODE by μ

$$y^{-4}\left(2xy\,dx + (y^2-3x^2)\,dy\right) = 0\cdot y^{-4}$$

$$2xy^{-3}\,dx + (y^{-2}-3x^2y^{-4})\,dy = 0$$

Check for exactness:

$$\frac{\partial(\mu M)}{\partial y} = -6xy^{-4} = \frac{\partial(\mu N)}{\partial x} = -6xy^{-4}$$

exact!

Solutions are given implicitly by $F(x,y) = C$ where

$$\frac{\partial F}{\partial x} = 2xy^{-3} \quad \text{and} \quad \frac{\partial F}{\partial y} = y^{-2} - 3x^2y^{-4}$$

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int 2xy^{-3}\,dx = x^2y^{-3} + g(y)$$

The "constant" of integration g could depend on y.

We need to find g. We know

$$\frac{\partial F}{\partial y} = y^{-2} - 3x^2 y^{-4}$$

From $F(x,y) = x^2 y^{-3} + g(y)$

$$\frac{\partial F}{\partial y} = -3x^2 y^{-4} + g'(y)$$

Matching

$$-3x^2 y^{-4} + g'(y) = y^{-2} - 3x^2 y^{-4}$$

$$g'(y) = y^{-2}$$

An antiderivative is $g(y) = \int y^{-2}\,dy = \frac{y^{-1}}{-1} = \frac{-1}{y}$

up to an added constant

$$F(x,y) = x^2 y^{-3} - \frac{1}{y}$$

$\boxed{\begin{array}{l}\text{The solutions to the ODE are implicitly} \\ \text{defined by} \\ \qquad x^2 y^{-3} - \frac{1}{y} = C\end{array}}$