February 4 Math 2306 sec 58 Spring 2016

Section 5: First Order Equations Models and Applications

Classic Mixing Problem: $r_i =$ inflow rate of fluid, $c_i =$ concentration of inflowing fluid, $r_o =$ outflow rate of fluid The concentration c_o of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We found the amount of salt at time t to be $A(t) = 1000 - 1000e^{-t/100}$ lbs, and the concentration at $t = 5$ to be about 0.098 lb/gal.

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

From before: $r_i = 5\ \frac{\text{gal}}{\text{min}}$, $c_i = 2\frac{\text{lb}}{\text{gal}}$

$A(0) = 0$

New info: $r_o = 10\ \frac{\text{gal}}{\text{min}}$

$$c_o = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}$$

$$c_o = \frac{A(t)}{500 + (5-10)t} = \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} = r_i c_i - r_o c_o = 5 \cdot 2 - 10 \cdot \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} + \frac{2}{100 - t} A = 10, \quad A(0) = 0$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

Current population P

Difference between M and P $\quad M-P$

$$\frac{dP}{dt} \propto P(M-P) \quad \text{i.e.} \quad \frac{dP}{dt} = kP(M-P) \text{ for}$$

some constant k.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

let $P(0) = P_0$

$$\frac{dP}{dt} = kP(M-P) \implies \frac{1}{P(M-P)}\frac{dP}{dt} = k$$

$$\int \frac{1}{P(M-P)}\,dP = \int k\,dt$$

[2]The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM\,dt = kMt + C$$

$$\ln|P| - \ln|M-P| = kMt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + C$$

$$e^{\ln\left|\frac{P}{M-P}\right|} = e^{kMt+C} = e^{C} e^{kMt}$$

Let $A = e^C$ or $A = -e^C$

$$\frac{P}{M-P} = Ae^{kMt}$$

Apply $P(0) = P_0$ $\qquad \frac{P_0}{M-P_0} = Ae^0 = A \Rightarrow \boxed{A = \frac{P_0}{M-P_0}}$

From $\frac{P}{M-P} = Ae^{kMt}$, mult. by $M-P$

$$P = Ae^{kMt}(M-P) = AMe^{kMt} - Ae^{kMt}P$$

$$P + Ae^{kMt}P = AMe^{kMt}$$

$$\left(1+Ae^{kMt}\right)P = AMe^{kMt}$$

$$P = \frac{AMe^{kMt}}{1+Ae^{kMt}}$$

Recall $A = \dfrac{P_0}{M-P_0}$

$$P(t) = \frac{\frac{P_0}{M-P_0}Me^{kMt}}{1+\frac{P_0}{M-P_0}e^{kMt}} \cdot \left(\frac{M-P_0}{M-P_0}\right)$$

$$\boxed{P(t) = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} \qquad \text{Solution to the IVP}}$$

Let's take $t \to \infty$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} = \text{"}\frac{\infty}{\infty}\text{"}$$

Use l'Hospital's rule

$$\lim_{t\to\infty} \frac{P_0 M kM \cancel{e^{kMt}}}{P_0 kM \cancel{e^{kMt}}}$$

$$= \lim_{t\to\infty} \frac{\cancel{P_0} M \cancel{k}\cancel{M}}{\cancel{P_0} \cancel{k} \cancel{M}}$$

$$= \lim_{t\to\infty} M = M$$