February 4 Math 2306 sec 58 Spring 2016

Section 5: First Order Equations Models and Applications

Classic Mixing Problem: r_i = inflow rate of fluid, c_i = concentration of inflowing fluid, r_o = outflow rate of fluid The concentration c_o of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt}=r_i\cdot c_i-r_o\frac{A}{V}.$$

February 2, 2016 1 / 57

This equation is first order linear.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt A(t) in pounds at the time t. Find the concentration of the mixture in the tank at t = 5 minutes.

We found the amount of salt at time *t* to be $A(t) = 1000 - 1000e^{-t/100}$ lbs, and the concentration at t = 5 to be about 0.098 lb/gal.

$$r_i \neq r_o$$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by A(t) under this new condition.

From before:
$$\Gamma_{i} = S \frac{gal}{min}$$
, $C_{i} = 2 \frac{15}{8al}$
 $A(0) = 0$
New info: $\Gamma_{o} = 10 \frac{gal}{min}$
 $C_{o} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_{i} - r_{o})t}$

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$$C_{o} = \frac{A(k)}{500 + (5 - 10)t} = \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} = r_i c_i - r_o c_o = 5.2 - 10. \frac{A}{s_{00} - st}$$

$$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$
, $A(0) = 0$

February 2, 2016 4 / 57

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A Nonlinear Modeling Problem

A population P(t) of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity¹ M of the environment and the current population. Determine the differential equation satsified by P.

Current population P
Difference between M and P M-P
$$\frac{dP}{dt} \propto P(M-P)$$
 i.e., $\frac{dP}{dt} = kP(M-P)$ for
some constant k

¹The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

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$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a logistic growth equation.

Solve this equation² and show that for any $P(0) \neq 0$, $P \rightarrow M$ as $t \rightarrow \infty$.

let P(0)=Po

$$\frac{dP}{dt} = kP(M-P) \implies \frac{1}{P(M-P)}\frac{dP}{dt} = k$$

$$\int \frac{1}{P(M-P)} dP = \int k dt$$

²The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M} \left(\frac{1}{P} + \frac{1}{M-P} \right)$$

is useful.

February 2, 2016 6 / 57

$$\int \frac{1}{M} \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int k dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int k M dt = k M t + C$$

$$\int n |P| - J_n |n-P| = k M t + C$$

$$\int n \left| \frac{P}{M-P} \right| = k M t + C$$

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February 2, 2016 7 / 57

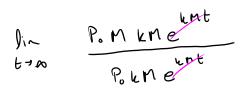
Let
$$A = e^{c}$$
 or $A = -e^{c}$
 $\frac{P}{m-P} = A e^{kMt}$
Apply $P(o) = P_{o}$ $\frac{P_{o}}{m-P_{o}} = A e^{o} = A \Rightarrow A = \frac{P_{o}}{m-P_{o}}$
From $\frac{P}{m-P} = A e^{kMt}$, mult, by $m-P$
 $P = A e^{kMt} (m-P) = A m e^{kMt} - A e^{kMt} P$
 $P + A e^{kMt} P = A m e^{kMt}$

$$(1 + Ae^{knt})P = AMe^{knt}$$
$$P = \frac{AMe^{kmt}}{1 + Ae^{kmt}}$$

Recall
$$A = \frac{P_o}{M - P_o}$$

 $P(t) = \frac{\frac{P_o}{M - P_o}}{1 + \frac{P_o}{M - P_o}} e^{knt} \cdot \left(\frac{N - P_o}{M - P_o}\right)$

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 February 2, 2016 9 / 57



$$=\lim_{E\to\infty}M=M$$

February 2, 2016 11 / 57

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