February 4 Math 2306 sec 59 Spring 2016

Section 5: First Order Equations Models and Applications

Classic Mixing Problem: $r_i =$ inflow rate of fluid, $c_i =$ concentration of inflowing fluid, $r_o =$ outflow rate of fluid The concentration c_o of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We found the amount of salt at time t to be $A(t) = 1000 - 1000e^{-t/100}$ lbs, and the concentration at $t = 5$ to be about 0.098 lb/gal.

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

From before : $r_i = 5 \frac{\text{gal}}{\text{min}}$, $c_i = 2 \frac{\text{lb}}{\text{gal}}$

New info : $r_o = 10 \frac{\text{gal}}{\text{min}}$

$$c_o = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t} = \frac{A(t)}{500 + (5-10)t}$$

$$\frac{dA}{dt} = 5 \cdot 2 - 10 \cdot \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} + \frac{2}{100 - t} A = 10 \quad , \quad A(0) = 0$$

This holds for $0 \le t < 100$

at $t = 100$ minutes, the tank

is empty.

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

Current Population is P

Difference between M and P is $M-P$

$\frac{dP}{dt} \propto P(M-P)$, i.e. $\frac{dP}{dt} = kP(M-P)$

for some constant k.

[1]The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

call $P(0) = P_0$

$$\frac{1}{P(M-P)} \frac{dP}{dt} = k \quad \Rightarrow \quad \int \frac{1}{P(M-P)}\, dP = \int k\, dt$$

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\, dt$$

[2] The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM\, dt$$

$$\ln|P| - \ln|M-P| = kMt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + C$$

$$e^{\ln\left|\frac{P}{M-P}\right|} = e^{kMt+C} = e^{C} e^{kMt}$$

Let $A = e^{C}$ or $A = -e^{C}$

$$\frac{P}{M-P} = A e^{kMt}$$

Use $P(0)=P_0$: $\frac{P_0}{M-P_0} = A e^{0} = A$

$$\boxed{\text{So} \quad A = \frac{P_0}{M-P_0}}$$

From $\frac{P}{M-P} = Ae^{kMt}$, mult. by $M-P$

$$P = Ae^{kMt}(M-P) = AMe^{kMt} - Ae^{kMt}P$$

$$P + Ae^{kMt} P = AMe^{kMt}$$

$$(1 + Ae^{kMt}) P = AMe^{kMt}$$

$$P(t) = \frac{AMe^{kMt}}{1 + Ae^{kMt}}$$

Recall $A = \dfrac{P_0}{M - P_0}$

$$P(t) = \frac{\frac{P_0}{M-P_0} M e^{kMt}}{1 + \frac{P_0}{M-P_0} e^{kMt}} \cdot \left(\frac{M-P_0}{M-P_0}\right)$$

$$\boxed{P(t) = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}}$$

Let's take $t \to \infty$

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} = \text{"}\frac{\infty}{\infty}\text{"}$$

Use l'Hospital's rule

$$\lim_{t \to \infty} \frac{\cancel{P_0} M \cancel{e^{kMt}} \cdot \cancel{kM}}{\cancel{P_0} \cancel{e^{kMt}} \cdot \cancel{kM}}$$

$$= \lim_{t \to \infty} M = M$$

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

to solve subject to conditions

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Example

Use only a little clever intuition to solve the IVP

$$y'' + 3y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$$

Homogeneous, $a_2(x) = \underline{1}$, $a_1(x) = 3$, $a_0(x) = -2$

$g(x) = 0$

all continuous $\underline{\underline{\text{and}}}$ $a_2(x) \neq 0$

Note that the constant function $y(x) = 0$ solves the IVP. By our theorem, this is $\underline{\underline{\text{the}}}$ only solution to the IVP.

A Second Order Linear Boundary Value Problem

consists of a problem

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad a < x < b$$

to solve subject to a pair of conditions[3]

$$y(a) = y_0, \quad y(b) = y_1.$$

However similar this is in appearance, the existence and uniqueness result **does not hold** for this BVP!

[3]Other conditions on y and/or y' can be imposed. The key characteristic is that conditions are imposed at both end points $x = a$ and $x = b$.