February 4 Math 2306 sec. 60 Spring 2019

Section 4: Exact Equations

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

Suppose that the equation $M\,dx + N\,dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x,y) = xy^2$.

$$xy^2(2y-6x)\,dx + xy^2(3x-4x^2y^{-1})\,dy = 0, \quad \Longrightarrow$$

$$\underbrace{(2xy^3-6x^2y^2)}_{M\mu}\,dx + \underbrace{(3x^2y^2-4x^3y)}_{N\mu}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial(M\mu)}{\partial y} &= 6xy^2 - 12x^2y \\ \frac{\partial(N\mu)}{\partial x} &= 6xy^2 - 12x^2y \end{aligned}\right\}\text{equal}$$

Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers n and m). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on x, or there is an integrating factor $\mu = \mu(y)$ depending only on y.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M\,dx + N\,dy = 0$$

is NOT exact, but that

$$\mu M\,dx + \mu N\,dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on y. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

Let's use the product rule in the right side.

Special Integrating Factor $\mu = \mu(x)$

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

$$\frac{\partial(\mu M)}{\partial y} = \underbrace{\frac{d\mu}{dy}}_{0} M + \mu \frac{\partial M}{\partial y}$$ product rule

$$= \mu \frac{\partial M}{\partial y}$$

$$\frac{\partial(\mu N)}{\partial x} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x}$$ product rule

These are equal so

$$\frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x} = \mu \frac{\partial M}{\partial y}$$

$$\frac{d\mu}{dx} N = \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x}$$

$$= \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \mu$$

Assuming $N(x,y) \neq 0$, we get

$$\frac{d\mu}{dx} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \mu$$

separable equation for μ

* This only makes sense (i.e. μ only exists) if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$ depends <u>only</u> on x

If this does only depend on x, we solve the equation for μ and set

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \, dx \right)$$

Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{1}$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is an special integrating factor for (1). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is an special integrating factor for (1).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

Check for exactness: $M(x,y) = 2xy$, $N(x,y) = y^2 - 3x^2$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = -6x \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

not exact

Look for μ as $\mu(x)$ or as $\mu(y)$

x: $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - (-6x)}{y^2 - 3x^2} = \frac{8x}{y^2 - 3x^2}$$

Depends on x and y

There is no $\mu(x)$ (function of only x)

$$y: \quad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x - 2x}{2xy} = \frac{-8x}{2xy} = \frac{-4}{y}$$

Depends only on y! There is an integrating factor $\mu = \mu(y)$

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\, dy\right) = e^{\int \frac{-4}{y}\, dy} = e^{-4\ln|y|} = y^{-4}$$

Use μ

$$y^{-4}\left(2xy\, dx + (y^2 - 3x^2)\, dy\right) = 0 \cdot y^{-4}$$

$$2xy^{-3}\,dx + \left(y^{-2} - 3x^2y^{-4}\right)dy = 0$$

Check to show it's exact now:

$$\frac{\partial(\mu M)}{\partial y} = -6xy^{-4} \qquad \frac{\partial(\mu N)}{\partial x} = -6xy^{-4}$$

It is exact. The solutions are $F(x,y) = C$ where

$$\frac{\partial F}{\partial x} = \mu M = 2xy^{-3} \quad \text{and}$$

$$\frac{\partial F}{\partial y} = \mu N = y^{-2} - 3x^2y^{-4}$$

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int 2xy^{-3}\,dx$$

$$= x^2 y^{-3} + g(y)$$

$g(y)$ is the "constant" of integration

To find g, use $\frac{\partial F}{\partial y}$: We know that

$$\frac{\partial F}{\partial y} = y^{-2} - 3x^2 y^{-4}$$

From F: $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y^{-3} + g(y)\right) = -3x^2 y^{-4} + g'(y)$

$$y^{-2} - 3x^2 y^{-4} = -3x^2 y^{-4} + g'(y)$$

$$g'(y) = y^{-2}$$

An antiderivative is $\int y^{-2}\,dy = \frac{y^{-1}}{-1} = \frac{-1}{y}$

Letting $g(y) = \frac{-1}{y}$, $F(x,y) = x^2y^{-3} - \frac{1}{y}$

(up to an added constant).

Solutions to the ODE are given implicitly by $F(x,y) = C$

$$\boxed{x^2y^{-3} - \frac{1}{y} = C}$$