# February 5 MATH 1112 sec. 54 Spring 2020

## Solving Exponential and Logarithmic Equations

**Base-Exponent Equality** For any $a > 0$ with $a \neq 1$, and for any real numbers $x$ and $y$

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

**Logarithm Equality** For and $a > 0$ with $a \neq 1$, and for any positive numbers $x$ and $y$

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

# Example

Solve the equation $3^{2x+1} = 81$. Since $81 = 3^4$

The equation is $3^{2x+1} = 3^4$.

So $2x+1 = 4$

$2x = 3$

$x = \frac{3}{2}$

Let's verify $3^{2x+1}$ set $x = \frac{3}{2}$

$3^{2(3/2)+1} = 3^{3+1} = 3^4 = 81$

# Example

Find an exact solution[1] to the equation

$2^{x+1} = 5^x$ We can use a log of any base.

Using the natural log:

$$\ln(2^{x+1}) = \ln(5^x)$$

$* \ln M^p = p \ln M$ for $M > 0$ and $p$ rational

$$(x+1)\ln 2 = x \ln 5$$

Now, we isolate $x$

$$x \ln 2 + \ln 2 = x \ln 5$$

[1] An exact solution may be a number such as $\sqrt{2}$ or $\ln(7)$ which requires a calculator to approximate as a decimal.

$$\ln 2 = x \ln 5 - x \ln 2$$

$$x(\ln 5 - \ln 2) = \ln 2$$

$$x = \frac{\ln 2}{\ln 5 - \ln 2}$$

# Graphical Solution to $2^{x+1} = 5^x$

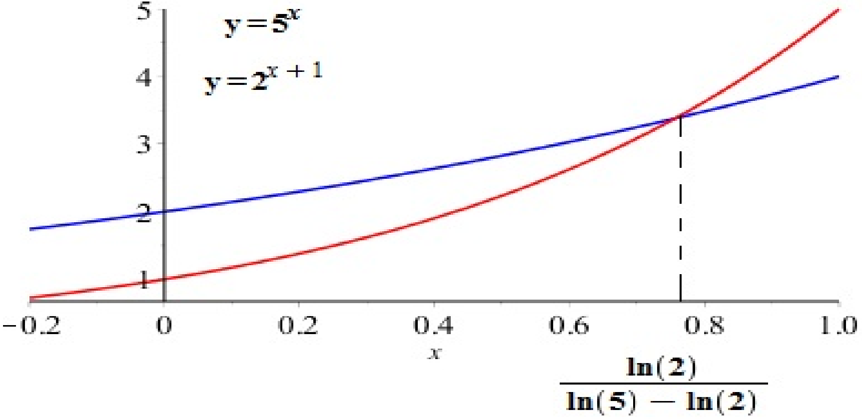


Figure: Plots of $y = 2^{x+1}$ and $y = 5^x$ together. The curves intersect at the solution $x = \ln 2/(\ln 5 - \ln 2) \approx 0.7565$.

**Which curve is $y = 2^{x+1}$, red or blue?**

# Log Equations & Verifying Answers

Double checking answers is always recommended. When dealing with functions whose domains are restricted, **answer verification** is critical.

Use properties of logarithms to solve the equation
$\log(x-1) + \log(x-2) = \log 12$

we'd like to get the form $\log(\star) = \log(\star\star)$

Recall $\log M + \log N = \log(MN)$ for $M, N > 0$

$\log(x-1) + \log(x-2) = \log 12$

$\log((x-1)(x-2)) = \log 12$

so $(x-1)(x-2) = 12$

Now, solve this quadratic equation

$$x^2 - 3x + 2 = 12$$

$$x^2 - 3x - 10 = 0$$

This factors nicely

$$(x-5)(x+2) = 0$$

$x - 5 = 0 \Rightarrow x = 5$ or

$x + 2 = 0 \Rightarrow x = -2$

We have to check whether these solve the original log equation.

$$\log(x-1) + \log(x-2) = \log 12$$

Check $x = 5$

$\log(5-1) + \log(5-2) =$

$\log 4 + \log 3 =$ $\Rightarrow$ 5 is a solution

$\log(4 \cdot 3) = \log 12$

Check $x = -2$

$\log(-2-1) + \log(-2-2)$

These are undefined -2 is not a solution.

# Applications

When a quantity $Q$ changes in time according to the model $Q(t) = Q_0 e^{kt}$, it is said to experience *exponential growth* ($k > 0$) or *exponential decay* ($k < 0$). Here, $Q_0$ is the constant initial quantity $Q(0)$, and $k$ is a constant. Examples of such phenomena include

- population changes (over small time frame),
- continuously compounded interest (on deposit or loan amount),
- substances subject to radio active decay

Generally, the model for exponential decay is written $Q(t) = Q_0 e^{-kt}$, and for growth it is written as $Q(t) = Q_0 e^{kt}$ so that it is always assumed that $k > 0$.

# Example

The $^{44}$Ti titanium isotope decays to $^{44}$Ca, a stable calcium isotope. The mass $Q$ is subject to exponential decay

$$Q(t) = Q_0 e^{-kt} \quad \text{for } t \text{ in years, and some} \quad k > 0.$$

If the half life (amount of time for the mass to reduce by 50%) is 60 years, determine the value of $k$.

We know that at time $t=0$ $Q(0) = Q_0$

and when $t=60$, $Q(60) = \frac{1}{2} Q_0$.

Since $Q(t) = Q_0 e^{-kt}$,

$$Q(60) = Q_0 e^{-k(60)} = Q_0 e^{-60k}$$

Setting the two forms of $Q(60)$ equal

$$\frac{1}{2} Q_0 = Q(60) = Q_0 e^{-60k}$$

$$Q_0 e^{-60k} = \frac{1}{2} Q_0$$ Divide by $Q_0$

$$e^{-60k} = \frac{1}{2}$$

$$\ln e^{-60k} = \ln \frac{1}{2}$$

$$-60k = \ln \frac{1}{2}$$

so $$k = \frac{\ln \frac{1}{2}}{-60}$$

we can rewrite this using

$$\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$$

so

$$k = \frac{-\ln 2}{-60} = \frac{\ln 2}{60}$$