February 5 Math 3260 sec. 51 Spring 2020

Section 1.9: The Matrix for a Linear Transformation

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

The matrix, called the **standard matrix** for the linear transformation T is given by

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)] \,.$$

One to one and Onto

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b} \quad \text{is always solvable.}$$

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$

Some Theorems on Onto and One to One

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

A Comment on Notation

Recall that a vector $\mathbf{x}$ in $\mathbb{R}^n$ is an ordered n-tuple. That is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \dots, x_n).$$

For example

$$(1,2,3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad (y_1, y_2, y_3, y_4) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix},$$

and

$$(x+y, 3z+u, 6w) = \begin{bmatrix} x+y \\ 3z+u \\ 6w \end{bmatrix}.$$

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

We already saw that

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = (1, 2, 0) \quad \text{and}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (0, -1, 3),$$

making the standard matrix for T

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}.$$

To show that T is one to one, we can show that $T(\vec{x}) = \vec{0}$ – i.e. $A\vec{x} = \vec{0}$ – has only the trivial solution

The augmented matrix for $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$$

It only has the trivial solution. Hence T is one to one.

Is T onto? We can ask if $T(\vec{x}) = \vec{b}$ is always solvable, i.e. is $A\vec{x} = \vec{b}$ always consistent?

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Then $A\vec{x} = \vec{b}$

has augmented matrix $\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix}$.

Doing a little row reduction

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & 6b_1 - 3b_2 - b_3 \end{bmatrix}$$

$A\vec{x} = \vec{b}$ is only consistent if $6b_1 - 3b_2 - b_3 = 0$.

Since $T(\vec{x}) = \vec{b}$ is not soluble for every $\vec{b}$ in $\mathbb{R}^3$, T is not onto.

To sum up:

The standard matrix $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$

Since $T: \mathbb{R}^2 \to \mathbb{R}^3$, A is 3×2.

Since $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

That's it. That's A. No extra columns, no x_1 or x_2, no rrefs. Just that.

To see if T is one to one, we can investigate the homogeneous equation $A\vec{x} = \vec{0}$.

If there's only the trivial solution → T is one to one.

If there are nontrivial solutions → T is not one to one.

We found that there was <u>only</u> the trivial solution.

So T is one to one.

To see if T is onto, we investigate $A\vec{x} = \vec{b}$ for arbitrary $\vec{b}$.

If $A\vec{x} = \vec{b}$ is always consistent → T is onto.

If $A\vec{x} = \vec{b}$ could be inconsistent → T is not onto.

We found that $A\vec{x} = \vec{b}$ is only consistent for some types of vectors $\vec{b}$ – not for <u>all</u> $\vec{b}$.

So T is not onto.