February 6 MATH 1112 sec. 54 Spring 2019

Section 5.3: Inverse of an Exponential Function

Definition: Let $a > 0$ and $a \neq 1$. For $x > 0$ define $\log_a(x)$ as a number such that

$$\text{if} \quad y = \log_a(x) \quad \text{then} \quad x = a^y.$$

The function

$$F(x) = \log_a(x)$$

is called the **logarithm function of base** a. It has domain $(0, \infty)$, range $(-\infty, \infty)$, and if $f(x) = a^x$ then

$$F(x) = f^{-1}(x).$$

In particular

- $\log_a(a^x) = x$ for every real x, and
- $a^{\log_a(x)} = x$ for every $x > 0$.

Graph of Logarithms

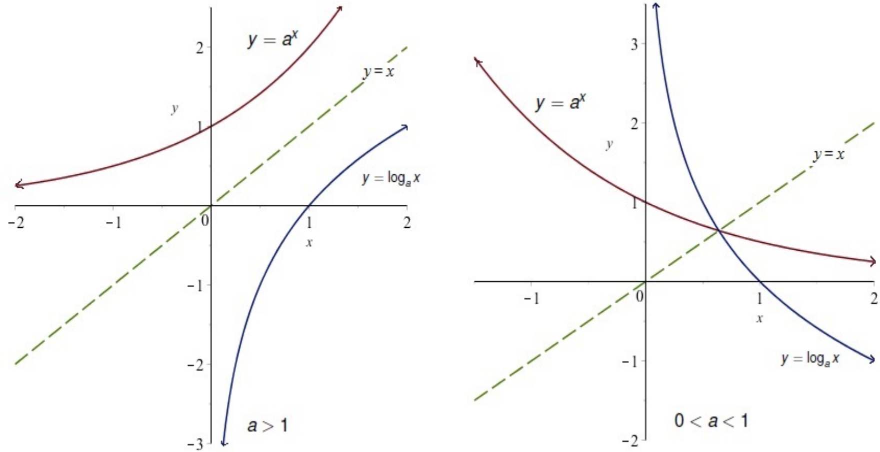

Figure: The graph of a logarithm can be obtained from the graph of an exponential by reflection in the line $y = x$. There are two cases depending on whether $0 < a < 1$ or $a > 1$.

Graph of Logarithms

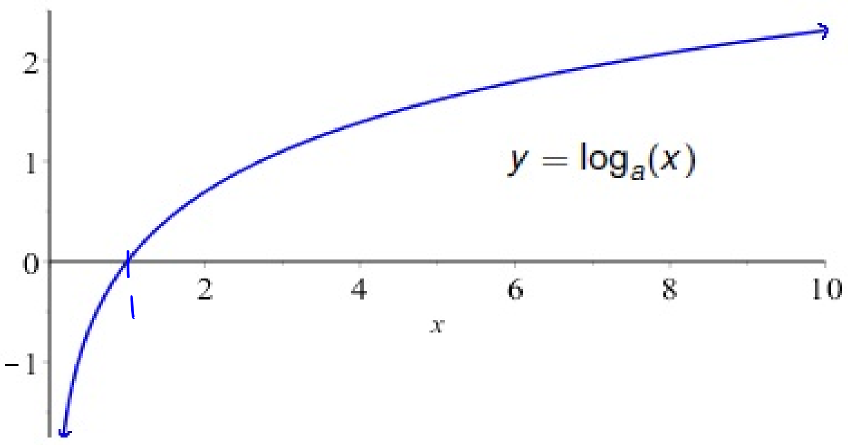

Figure: The graph of a logarithm with a base a where $a > 1$.

Graph of Logarithms

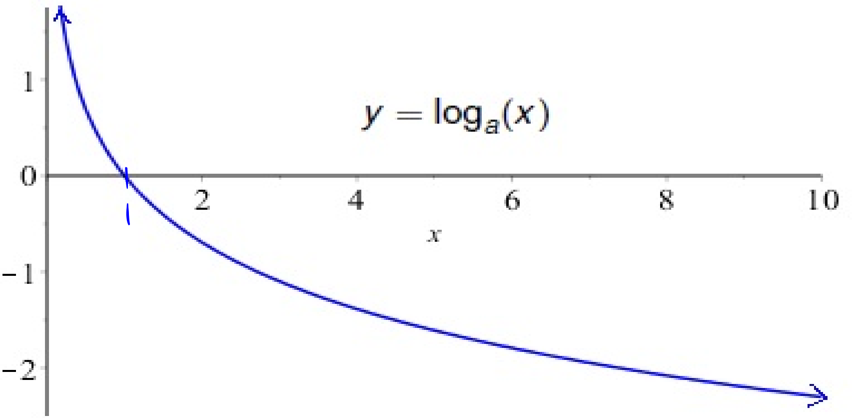

Figure: The graph of a logarithm with a base a where $0 < a < 1$

Evaluating Simple Logarithms

Use the fact that $y = \log_a(x)$ means $x = a^y$ to evaluate

(a) $\log_2(16)$ $= 4$ $\qquad 2^y = 16$

(b) $\log_{10}(0.001)$ $= -3$ $\qquad 10^y = 0.001$

(c) $\log_{1/2}(4)$ $= -2$ $\qquad \left(\frac{1}{2}\right)^y = 4$

(d) $\log_a(a^7)$ $= 7$ $\qquad a^y = a^7$

(e) $\log_\pi(1)$ $= 0$ $\qquad \pi^y = 1$

Question

Recall: $y = \log_a(x)$ means $x = a^y$.

If $e^t = 70$, then (which statement is true)

(a) $\log_e(t) = 70$

(b) $\log_{70}(e) = t$

(c) $\log_e(70) = t$

(d) $\log_t(e) = 70$

A Few Properties

- For every $a > 0$, $a^0 = 1$, hence $\log_a(1) = 0$.
- For every $a > 0$, $a^1 = a$, hence $\log_a(a) = 1$.
- For every $a > 0$, the expression $\log_a(0)$ is UNDEFINED!

Graphically: Every graph $y = \log_a(x)$ passes through the points $(1, 0)$ and $(a, 1)$. And

$$\begin{array}{llll} \text{if } a > 1 & \log_a(x) \to -\infty & \text{as} & x \to 0^+ \\ \text{if } 0 < a < 1 & \log_a(x) \to \infty & \text{as} & x \to 0^+ \end{array}$$

Evaluating Logs with a Calculator

Most logarithms can't be evaluated by hand. For example, $\log_e(70) \approx 4.2485$. No one memorizes that! Calculators tend to have two built in logarithm functions.

- **Common Log** base 10 denoted as log (note there is no subscript), and
- **Natural Log** base *e* denoted[1] ln

In Calculus, you'll find that the prefered base is *e*—the natural log. Base 10 logs still have some use such as in the Richter scale used to measure earthquakes.

[1] The order LN instead of NL is probably due to the French name *le Logarithme Naturel* for this log.

Evaluating Logs with a Calculator

Suppose we wish to evaluate $\log_2(15)$. You turn to a calculator, but there is no $\log_2$ key! Fortunately, you're not out of luck. Every log can be stated in term of any other log by the following theorem:

Theorem: (Change of Base) Let a, b, and M be any positive numbers, then

$$\log_b(M) = \frac{\log_a(M)}{\log_a(b)}.$$

$a \neq 1$

$b \neq 1$

What this says is that you can turn your $\log_2$ problem into a log or ln problem, and use your machine!

Change of Base

Here's the meat of our theorem again: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$.

Express $\log_2(15)$ in terms of the natural log. Use a calculator to approximate its value.

$$\log_2(15) = \frac{\ln(15)}{\ln(2)} \approx 3.9069$$

Question

Here's the meat of our theorem again: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$.

Suppose you wish to evaluate $\log_\pi(42)$ using a calculator. Which of the following expressions could be used to find the desired value?

(a) $\dfrac{\log(42)}{\log(\pi)}$

(b) $\ln\left(\dfrac{42}{\pi}\right)$

(c) $\dfrac{\ln(\pi)}{\ln(42)}$

(d) $\ln\left(\dfrac{\pi}{42}\right)$

Section 5.4: Properties of Logarithms

Let's recall the basic properties of exponentials. The logarithms have analog properties.

Let $a > 0$, with $a \neq 1$. Then for any real x and y

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$

Log of a Product

Theorem: Let M and N be any positive numbers and $a > 0$ with $a \neq 1$. Then

$$\log_a(MN) = \log_a(M) + \log_a(N).$$

Illustrative Example:

$$\log_2(16) = \log_2(2 \cdot 8) = \log_2(2) + \log_2(8).$$

Note that this equation is the true statement

$$4 = 1 + 3.$$