February 4 Math 2306 sec. 53 Spring 2019

Section 5: First Order Equations Models and Applications

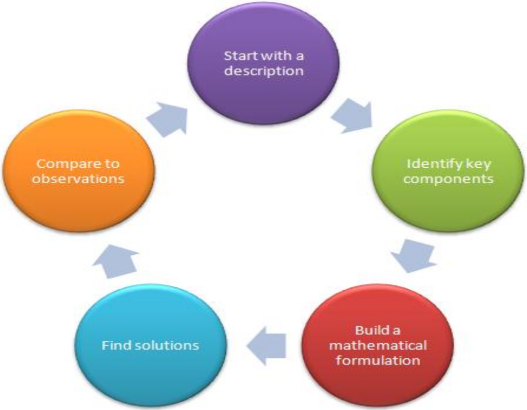


Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let $P(t)$ be the population density of rabbits (no. individuals per unit habitat) at time t in years since 2011.

The rate of change of P is $\frac{dP}{dt}$. $\frac{dP}{dt}$ is proportional to P means $\frac{dP}{dt} = kP$ for some constant k.

This is a 1st order linear and separable ODE for P.

The information on population counts in 2011 and 2012 can be expressed as

$P(0) = 58$ and $P(1) = 89$

Together $\frac{dP}{dt} = kP$, $P(0) = 58$ is an IVP.

Solving using separation of variables

$$\frac{1}{P}\frac{dP}{dt} = k$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + C \qquad \text{note } P > 0$$

$$P = e^{kt+C} = Ae^{kt} \quad \text{where } A = e^{C}$$

Apply $P(0) = 58$ to get

$$P(0) = Ae^0 = 58 \implies A = 58.$$

Hence

$$P(t) = 58e^{kt}.$$

To find k, we'll use $P(1) = 89$

$$P(1) = 58e^{k(1)} = 89$$

$$e^k = \frac{89}{58}$$

$$k = \ln\left(\frac{89}{58}\right)$$

So the population function

$$P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}$$

In 2021, $t = 10$. The population is expected to be

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4198$$

Roughly 4200 rabbits.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

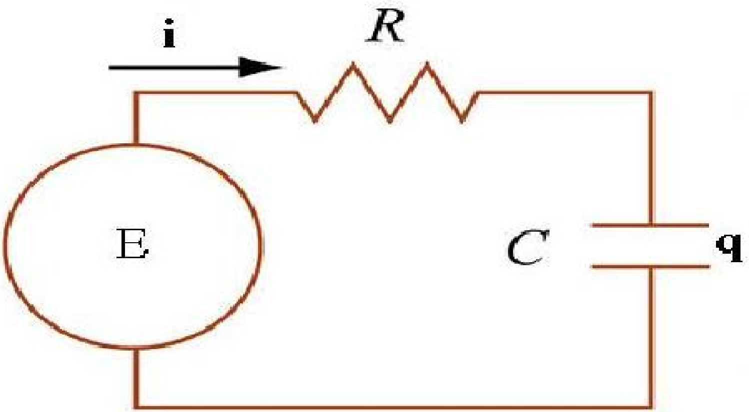


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

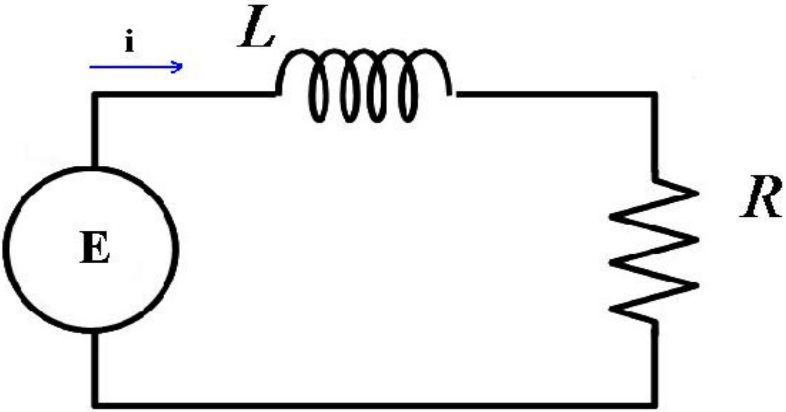


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),
Inductance L in henries (h),
Capacitance C in farads (f),
Implied voltage E in volts (V),
Charge q in coulombs (C),
Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \frac{dq}{dt}$.

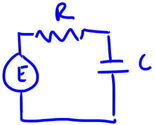
Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC



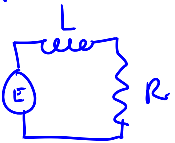
Potential Drops = Implied force

across resistor + across capacitor = E

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

1st order Linear ODE for charge q.

LR



across inductor + across resistor = E

$$L\frac{di}{dt} + Ri = E$$

1st order Linear ODE for current i.