Section 5: First Order Equations Models and Applications

We considered the following problem description:

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We defined $P(t)$ as the population density (no. rabbits per unit habitat) at the time $t$ in years with $t = 0$ in 2011. The given statement was then interpreted mathematically as

$$\frac{dP}{dt} = kP, \quad P(0) = 58, \quad \text{and} \quad P(1) = 89.$$
\[ \frac{dP}{dt} = kP, \quad P(0) = 58 \]

With the first condition, we have an IVP for the population of rabbits \( P \).
Exponential Growth or Decay

If a quantity $P$ changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$ 

Note that this equation is both separable and first order linear. If $k > 0$, $P$ experiences exponential growth. If $k < 0$, then $P$ experiences exponential decay.
Series Circuits: RC-circuit

Figure: Series Circuit with Applied Electromotive force $E$, Resistance $R$, and Capacitance $C$. The charge of the capacitor is $q$ and the current $i = \frac{dq}{dt}$. 
Series Circuits: LR-circuit

Figure: Series Circuit with Applied Electromotive force $E$, Inductance $L$, and Resistance $R$. The current is $i$. 
Measurable Quantities:

Resistance $R$ in ohms ($\Omega$),
Inductance $L$ in henries (h),
Capacitance $C$ in farads (f),
Implied voltage $E$ in volts (V),
Charge $q$ in coulombs (C),
Current $i$ in amperes (A)

Current is the rate of change of charge with respect to time: $i = \frac{dq}{dt}$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Potential Drop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductor</td>
<td>$L \frac{di}{dt}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$Ri$ i.e. $R \frac{dq}{dt}$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$\frac{1}{C}q$</td>
</tr>
</tbody>
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Kirchhoff’s Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.
Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance $5 \times 10^{-6}$ f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4\text{A}$. Determine the charge as $t \to \infty$. 
A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time $t$. Find the concentration of the mixture in the tank at $t = 5$ minutes.
A Classic Mixing Problem

**Figure:** Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.
Building an Equation

The rate of change of the amount of salt

\[
\frac{dA}{dt} = \left( \text{input rate of salt} \right) - \left( \text{output rate of salt} \right)
\]

The input rate of salt is

\[
\text{fluid rate in} \cdot \text{concentration of inflow} = r_i(c_i).
\]

The output rate of salt is

\[
\text{fluid rate out} \cdot \text{concentration of outflow} = r_o(c_o).
\]
Building an Equation

The concentration of the outflowing fluid is

\[
\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.
\]

\[
\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.
\]

This equation is first order linear.
Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt \( A(t) \) in pounds at the time \( t \). Find the concentration of the mixture in the tank at \( t = 5 \) minutes.
\[ r_i \neq r_o \]

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by \( A(t) \) under this new condition.
A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity $M$ of the environment and the current population. Determine the differential equation satisfied by $P$.

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$^1$The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.
Logistic Differential Equation

The equation

\[
\frac{dP}{dt} = kP(M - P), \quad k, M > 0
\]

is called a logistic growth equation.

Solve this equation\(^2\) and show that for any \(P(0) \neq 0\), \(P \to M\) as \(t \to \infty\).

\(^2\)The partial fraction decomposition

\[
\frac{1}{P(M - P)} = \frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right)
\]

is useful.
Recall that an $n^{th}$ order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$ 

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.
Theorem: Existence & Uniqueness

**Theorem:** If $a_0, \ldots, a_n$ and $g$ are continuous on an interval $I$, $a_n(x) \neq 0$ for each $x$ in $I$, and $x_0$ is any point in $I$, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on $I$.

Put differently, we’re guaranteed to have a solution exist, and it is the only one there is!
Homogeneous Equations

We’ll consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

and assume that each \( a_i \) is continuous and \( a_n \) is never zero on the interval of interest.

**Theorem:** If \( y_1, y_2, \ldots, y_k \) are all solutions of this homogeneous equation on an interval \( I \), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \( I \) for any choice of constants \( c_1, \ldots, c_k \).

This is called the principle of superposition.
Corollaries

(i) If $y_1$ solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any nontrivial solution(s), and
- since $y_1$ and $cy_1$ aren’t truly different solutions, what criteria will be used to call solutions distinct?
Linear Dependence

**Definition:** A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval $I$ if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$ 

A set of functions that is not linearly dependent on $I$ is said to be **linearly independent** on $I$. 
Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.
Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$. 
Determine if the set is Linearly Dependent or Independent on \((-\infty, \infty)\)

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]
Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ possess at least $n - 1$ continuous derivatives on an interval $I$. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$ 

(Note that, in general, this Wronskian is a function of the independent variable $x$.)
Determinants

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If $A$ is a $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$
Determine the Wronskian of the Functions

\[ f_1(x) = \sin x, \quad f_2(x) = \cos x \]
Determine the Wronskian of the Functions

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]
Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n - 1$ times continuously differentiable on an interval $I$. If there exists $x_0$ in $I$ such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are \textbf{linearly independent} on $I$.

If $y_1, y_2, \ldots, y_n$ are $n$ solutions of the linear homogeneous $n^{th}$ order equation on an interval $I$, then the solutions are \textbf{linearly independent} on $I$ if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for each $x$ in $I$.

\footnote{For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.}
Determine if the functions are linearly dependent or independent:

\[ y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty) \]
**Fundamental Solution Set**

We’re still considering this equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0 \]

with the assumptions \( a_n(x) \neq 0 \) and \( a_i(x) \) are continuous on \( I \).

**Definition:** A set of functions \( y_1, y_2, \ldots, y_n \) is a fundamental solution set of the \( n^{th} \) order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are \( n \) of them, and

(iii) they are linearly independent.

**Theorem:** Under the assumed conditions, the equation has a fundamental solution set.
Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the $n^{th}$ order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
Example
Verify that \( y_1 = e^x \) and \( y_2 = e^{-x} \) form a fundamental solution set of the ODE

\[
y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),
\]
and determine the general solution.
Nonhomogeneous Equations

Now we will consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \]

where \( g \) is not the zero function. We’ll continue to assume that \( a_n \) doesn’t vanish and that \( a_i \) and \( g \) are continuous.

The **associated homogeneous equation** is

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \]
Theorem: General Solution of Nonhomogeneous Equation

Let $y_p$ be any solution of the nonhomogeneous equation, and let $y_1, y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)
Another Superposition Principle (for nonhomogeneous eqns.)

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be $k$ particular solutions to the nonhomogeneous linear equations

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all $k$ equations is a common interval $I$.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$
Example \( x^2y'' - 4xy' + 6y = 36 - 14x \)

(a) Verify that

\[
y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.
\]
Example $x^2 y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2 y'' - 4xy' + 6y = -14x.$$
Example $x^2 y'' - 4xy' + 6y = 36 - 14x$

(c) It is readily shown that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2 y'' - 4xy' + 6y = 0.$$ 

Use this along with results (a) and (b) to write the general solution of $x^2 y'' - 4xy' + 6y = 36 - 14x$. 
Solve the IVP

\[ x^2 y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5 \]