Section 5: First Order Equations Models and Applications

We considered the following problem description:

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We defined $P(t)$ as the population density (no. rabbits per unit habitat) at the time $t$ in years with $t = 0$ in 2011. The given statement was then interpreted mathematically as

$$\frac{dP}{dt} = kP, \quad P(0) = 58, \quad \text{and} \quad P(1) = 89.$$
\[
\frac{dP}{dt} = kP, \quad P(0) = 58
\]

With the first condition, we have an IVP for the population of rabbits \( P \).

The ODE is separable (it's also 1st order linear).

\[
\frac{dP}{dt} = kP \implies \frac{1}{P} \frac{dP}{dt} = k
\]

\[
\int \frac{1}{P} \, dP = \int k \, dt
\]

\[
\ln P = kt + C \implies P = e^{kt+C} = A e^{kt} \quad \text{where} \quad A = e^C
\]

\[
P(0) = 58 \quad \text{so} \quad P(0) = A e^0 = 58
\]
A = 58, so \[ P(t) = 58 e^{kt}. \]

From \( P(1) = 89 \), we can find \( k \).

\[ P(1) = 89 = 58 e^k \quad \Rightarrow \quad e^k = \frac{89}{58} \quad \Rightarrow \quad k = \ln \left( \frac{89}{58} \right) \]

Hence \[ P(t) = 58 e^{t\ln \left( \frac{89}{58} \right)} \]

In 2021, \( t = 10 \). The population is approximately \[ P(10) = 58 e^{10\ln \left( \frac{89}{58} \right)} \approx 4197. \]
Exponential Growth or Decay

If a quantity $P$ changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \text{ i.e. } \frac{dP}{dt} - kP = 0.$$ 

Note that this equation is both separable and first order linear. If $k > 0$, $P$ experiences **exponential growth**. If $k < 0$, then $P$ experiences **exponential decay**.
Series Circuits: RC-circuit

Figure: Series Circuit with Applied Electromotive force $E$, Resistance $R$, and Capacitance $C$. The charge of the capacitor is $q$ and the current $i = \frac{dq}{dt}$. 
Series Circuits: LR-circuit

Figure: Series Circuit with Applied Electromotive force $E$, Inductance $L$, and Resistance $R$. The current is $i$. 
Measurable Quantities:

Resistance $R$ in ohms ($\Omega$), Implied voltage $E$ in volts (V),
Inductance $L$ in henries (h), Charge $q$ in coulombs (C),
Capacitance $C$ in farads (f), Current $i$ in amperes (A)

Current is the rate of change of charge with respect to time: $i = \frac{dq}{dt}$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Potential Drop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductor</td>
<td>$L \frac{di}{dt}$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$Ri$ i.e. $R \frac{dq}{dt}$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$\frac{1}{C}q$</td>
</tr>
</tbody>
</table>
Kirchhoff’s Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

For the RC:  

\[ R \frac{dq}{dt} + \frac{1}{C} q = E \]

1st order linear ODE for \( q(t) \).
For the LR: Potential Drop across

\[ L \frac{di}{dt} + Ri = E \]

1st order linear ODE for \( i(t) \).
Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance $5 \times 10^{-6}$ f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4$ A. Determine the charge as $t \to \infty$.

\[ R \frac{dq}{dt} + \frac{1}{C} q = E \]

\[ E = 200 \text{ V} \quad C = 5 \times 10^{-6} \text{ f} \]

\[ R = 1000 \Omega \]

\[ q'(0) = i(0) = 0.4 \text{ A} \]

\[ 1000 \frac{dq}{dt} + \frac{1}{5 \times 10^{-6}} q = 200 \]

\[ \frac{10^6}{5(1000)} = \frac{10^3}{5} = 200 \]

\[ \frac{dq}{dt} + \frac{1}{5 \times 10^{-6} \cdot 1000} q = \frac{200}{1000} \]
\[ \frac{d}{dt} + 200 \ q = \frac{1}{5} \]  
\[ P(t) = 200 \]
\[ \mu = e \]
\[ \int P(t) \ dt \]
\[ \int 200 \ dt \]
\[ \int 200 \ dt \]

\[ \frac{d}{dt} \left[ e^{200t} \ q \right] = \frac{1}{5} \ e^{200t} \]

\[ \int \frac{1}{5} \ e^{200t} \ dt \]

\[ e^{200t} \ q = \frac{1}{5} \ e^{200t} + K \]

\[ q = \frac{1}{1000} + K \ e^{-200t} \]
Apply \( i(0) = 0.4 \) i.e. \( q'(0) = 0.4 \)

\[
q'(t) = k(-200 e^{-200t}) = -200 ke^{-200t}
\]

\[
q'(0) = 0.4 = -200 ke^0 = -200k
\]

\[
k = \frac{0.4}{-200} = -0.002
\]

So the charge \( q \) on the capacitor is

\[
q(t) = 0.001 - 0.002 e^{-200t}
\]

The long time charge

\[
\lim_{{t \to \infty}} q(t) = \lim_{{t \to \infty}} (0.001 - 0.002 e^{-200t}) = 0.001 \text{ C}
\]
A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time $t$. Find the concentration of the mixture in the tank at $t = 5$ minutes.
A Classic Mixing Problem

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

\[ r \] - rate of fluid flow
\[ c \] - concentration
\[ i \] - in
\[ o \] - out

\[ \text{input rate } r_i \]
\[ \text{input concentration } c_i \]

\[ \text{output rate } r_o \]
\[ \text{output concentration } c_o \]
Building an Equation

The rate of change of the amount of salt

\[
\frac{dA}{dt} = \left( \text{input rate of salt} \right) - \left( \text{output rate of salt} \right)
\]

The input rate of salt is

\[
\text{fluid rate in} \cdot \text{concentration of inflow} = r_i(c_i).
\]

The output rate of salt is

\[
\text{fluid rate out} \cdot \text{concentration of outflow} = r_o(c_o).
\]
Building an Equation

\[ C_0 = \text{concentration in the tank} \]

The concentration of the outflowing fluid is

\[
\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.
\]

\[
\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.
\]

This equation is first order linear.

\[
\frac{dA}{dt} + \frac{C_0}{V} A = r_i c_i c_i
\]
Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time $t$. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500 \text{ gal}$

$r_i = 5 \frac{\text{gal}}{\text{min}}$

$c_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_o = 5 \frac{\text{gal}}{\text{min}}$, $A(0) = 0$ (pure water)

$C_0 = \frac{A}{V} = \frac{A \text{ lb}}{500 \text{ gal}}$

$V(t) = V(0) + (r_i - r_o) t = 500 \text{ gal} + (5 \frac{\text{gal}}{\text{min}} - 5 \frac{\text{gal}}{\text{min}}) \cdot t \text{ min}$

$= 500 \text{ gal}$
\[
\frac{dA}{dt} + \frac{r_0}{V} A = r_i \cdot C_i
\]

\[
\frac{dA}{dt} + \frac{s}{500} A = s(2) = 10
\]

\[
\frac{dA}{dt} + \frac{1}{100} A = 10 \quad P(t) = \frac{1}{100} \quad \mu = e^{-\frac{1}{100} t}
\]

\[
\frac{d}{dt} \left[ e^{\frac{1}{100} t} A \right] = 10 e^{\frac{1}{100} t}
\]

\[
\int \frac{d}{dt} \left[ e^{\frac{1}{100} t} A \right] dt = \int 10 e^{\frac{1}{100} t} dt
\]
\[ e^{\frac{t}{100}} A = 10 \left( 100 \right) e^{\frac{t}{100}} + k \]

\[ -\frac{1}{100} t \]

\[ A = 1000 + ke^{\frac{t}{100}} \]

**Apply** \( A(0) = 0 \)

\[ A(0) = 0 = 1000 + ke^{0} = 1000 + k \]

\[ k = -1000 \]

The amount of salt at time \( t \) in the tank is
\[ A(t) = 1000 - 1000e^{-\frac{t}{100}} \]

Letting \( C(t) \) be the concentration of salt in the tank at time \( t \):

\[ C(t) = \frac{A(t)}{V(t)} = \frac{1000 - 1000e^{-\frac{t}{100}}}{500} \text{ lb} \text{ per gal} \]

So,

\[ C(5) = \frac{1000 - 1000e^{-\frac{0.5}{100}}}{500} \approx 0.098 \text{ lb per gal} \]
Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$$V(0) = 500, \quad r_i = 5, \quad r_o = 10, \quad c_i = 2$$

$$c_0 = \frac{A}{V}$$

Now

$$V = V(0) + (r_i - r_o)t = 500 + (5 - 10)t$$

$$= 500 - 5t$$

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

$$\frac{dA}{dt} + \frac{10}{500-5t} A = 10 \quad \Rightarrow$$

$$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$

valid for $0 < t < 100$