February 6 Math 2306 sec. 60 Spring 2018

Section 5: First Order Equations Models and Applications

We considered the following problem description:

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We defined $P(t)$ as the population density (no. rabbits per unit habitat) at the time t in years with $t = 0$ in 2011. The given statement was then interpreted mathematically as

$$\frac{dP}{dt} = kP, \quad P(0) = 58, \quad \text{and} \quad P(1) = 89.$$

$\frac{dP}{dt} = kP, \quad P(0) = 58$

With the first condition, we have an IVP for the population of rabbits P.

The ODE is separable (it's also 1st order linear)

$$\frac{dP}{dt} = kP \implies \frac{1}{P}\frac{dP}{dt} = k$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln P = kt + C \implies P = e^{kt+C}$$

$$= Ae^{kt} \quad \text{where } A = e^{C}$$

$P(0) = 58$ so $P(0) = Ae^{0} = 58$

$A = 58$, so $P(t) = 58e^{kt}$

From $P(1) = 89$, we can find k.

$$P(1) = 89 = 58e^{k\cdot 1} \Rightarrow e^k = \frac{89}{58} \Rightarrow k = \ln\left(\frac{89}{58}\right)$$

Hence $P(t) = 58e^{t\ln\left(\frac{89}{58}\right)}$

In 2021, $t = 10$. The population is approximately

$$P(10) = 58e^{10\ln\left(\frac{89}{58}\right)} \approx 4198$$

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

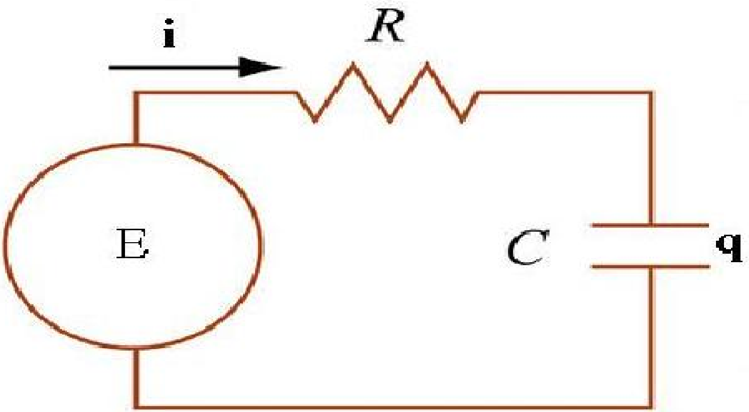


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

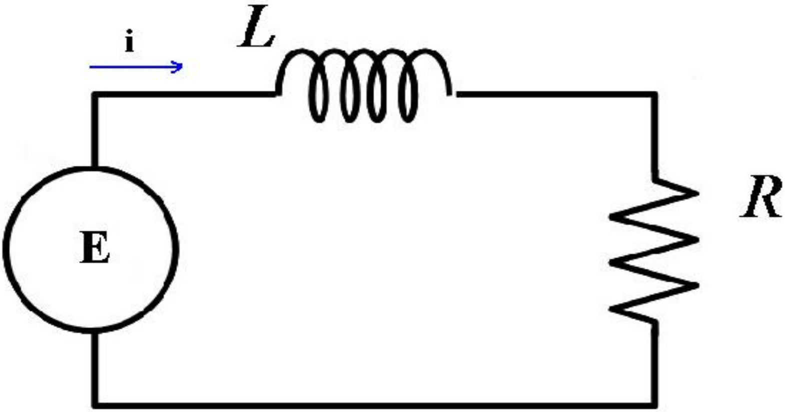


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Implied voltage E in volts (V),
Inductance L in henries (h), Charge q in coulombs (C),
Capacitance C in farads (f), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

For the RC:

Potential Drop across

Resistor Capacitor

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

1st order linear ODE for $q(t)$.

For the LR:

Potential Drop across

Inductor Resistor

$$L\frac{di}{dt} + Ri = E$$

1st order linear ODE for $i(t)$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0)=0.4A$. Determine the charge as $t\to\infty$.

$$R\frac{dq}{dt}+\frac{1}{C}q=E$$

$E=200$ V $\qquad C=5\cdot 10^{-6}$ f

$R=1000\,\Omega \qquad q'(0)=i(0)=0.4$ A

$$1000\frac{dq}{dt}+\frac{1}{5\cdot 10^{-6}}q=200$$

$$\frac{10^6}{5(1000)}=\frac{10^3}{5}=200$$

$$\frac{dq}{dt}+\frac{\frac{1}{5\cdot 10^{-6}}}{1000}q=\frac{200}{1000}$$

$$\frac{dq}{dt} + 200q = \frac{1}{5} \qquad P(t) = 200$$

$$\mu = e^{\int P(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left[e^{200t} q\right] = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left[e^{200t} q\right] dt = \int \frac{1}{5} e^{200t}\, dt$$

$$e^{200t} q = \frac{1}{5} \cdot \frac{1}{200} e^{200t} + K$$

$$q = \frac{1}{1000} + K e^{-200t}$$

Apply $i(0) = 0.4$ i.e. $q'(0) = 0.4$

$$q'(t) = k\left(-200e^{-200t}\right) = -200ke^{-200t}$$

$$q'(0) = 0.4 = -200ke^{0} = -200k$$

$$k = \frac{0.4}{-200} = -0.002$$

So the charge q on the capacitor is

$$q(t) = 0.001 - 0.002e^{-200t}.$$

The long time charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(0.001 - 0.002e^{-200t}\right) = 0.001 \text{ C}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

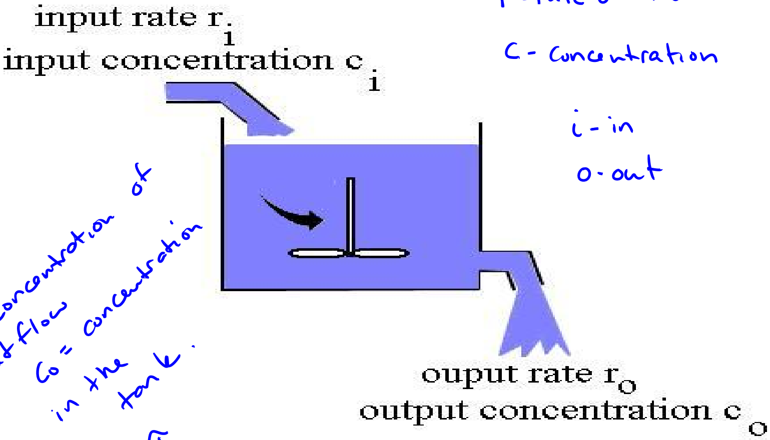
A Classic Mixing Problem

r - rate of fluid flow

c - concentration

i - in

o - out



Concentration of outflow

c_o = concentration in the tank.

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array}\right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

C_o = concentration in the tank

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

initial volume of fluid

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500$ gal $\qquad r_o = 5 \frac{\text{gal}}{\text{min}}$, $\quad A(0) = 0$ (pure water)

$r_i = 5 \frac{\text{gal}}{\text{min}}$ $\qquad c_o = \frac{A}{V} = \frac{A \text{ lb}}{500 \text{ gal}}$

$c_i = 2 \frac{\text{lb}}{\text{gal}}$

$V(t) = V(0) + (r_i - r_o)t = 500 \text{ gal} + \left(5 \frac{\text{gal}}{\text{min}} - 5 \frac{\text{gal}}{\text{min}}\right) \cdot t \text{ min}$

$= 500 \text{ gal}$

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i \cdot c_i$$

$$\frac{dA}{dt} + \frac{5}{500} A = 5(2) = 10$$

$$\frac{dA}{dt} + \frac{1}{100} A = 10 \qquad P(t) = \frac{1}{100} \qquad \mu = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] = 10 e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t} A\right) dt = \int 10 e^{\frac{1}{100}t} dt$$

$$e^{\frac{1}{100}t} A = 10(100)e^{\frac{1}{100}t} + k$$

$$A = 1000 + ke^{-\frac{1}{100}t}$$

Apply $A(0) = 0$

$$A(0) = 0 = 1000 + ke^{0} = 1000 + k$$

$$k = -1000$$

The amount of salt at time t in the tank is

$$A(t) = 1000 - 1000\,e^{-\frac{1}{100}t}$$

Letting $C(t)$ be the concentration of salt in the tank @ time t

$$C(t) = \frac{A(t)}{V(t)} = \frac{1000 - 1000\,e^{-\frac{1}{100}t}}{500} \quad \frac{\text{lb}}{\text{gal}}$$

so

$$C(5) = \frac{1000 - 1000\,e^{-\frac{1}{100}\cdot 5}}{500} \approx 0.098 \quad \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$V(0) = 500$, $r_i = 5$, $r_o = 10$, $c_i = 2$

$$c_o = \frac{A}{V}$$

Now $V = V(0) + (r_i - r_o)t = 500 + (5-10)t$

$= 500 - 5t$

$$\frac{dA}{dt} + \frac{r_o}{V}A = r_i c_i$$

$$\frac{dA}{dt} + \frac{10}{500-5t}A = 10 \implies \boxed{\frac{dA}{dt} + \frac{2}{100-t}A = 10}$$

valid for $0 < t < 100$