February 4 Math 2306 sec. 60 Spring 2019

Section 5: First Order Equations Models and Applications

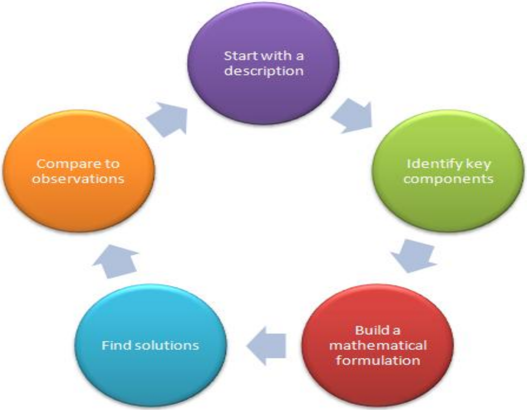


Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let $P(t)$ be the population density of rabbits (no. individuals per unit habitat) at time t in years since 2011.

Rate of change of P is $\frac{dP}{dt}$. $\frac{dP}{dt}$ proportional to P means

$$\frac{dP}{dt} = kP \quad \text{for some constant } k.$$

This is a differential equation for P. The population info given tells us that $P(0) = 58$ and $P(1) = 89$

* $t=0$ in 2011 and $t=1$ in 2012

Combining the first with the ODE, we have an IVP for P

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

Using separation of variables

$$\frac{1}{P}\frac{dP}{dt} = k \implies \int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + C$$

$$P = e^{kt+C} = Ae^{kt} \quad \text{where } A = e^{C}$$

Applying $P(0)=58$, $P(0)=Ae^{0}=58 \Rightarrow A=58$

so the population

$$P(t)=58e^{kt}$$

we can find k from $P(1)=89$

$$P(1)=58e^{k(1)}=89$$

$$e^{k}=\frac{89}{58}$$

$$k=\ln\left(\frac{89}{58}\right)$$

The population function is completely determined by

$$P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}$$

In 2021, $t = 10$. The population is approximated by

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4198$$

About 4200 rabbits.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

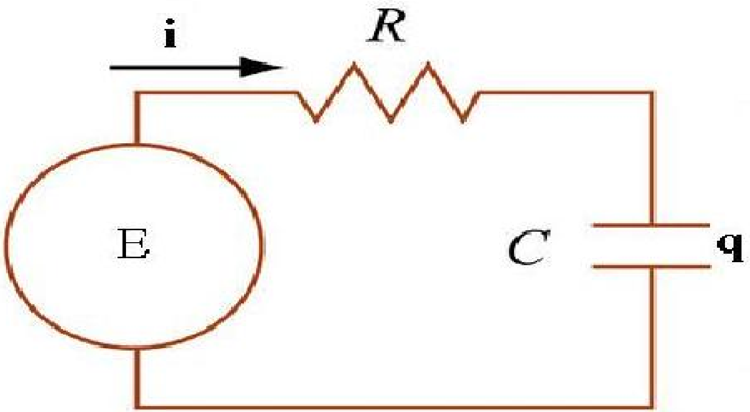


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

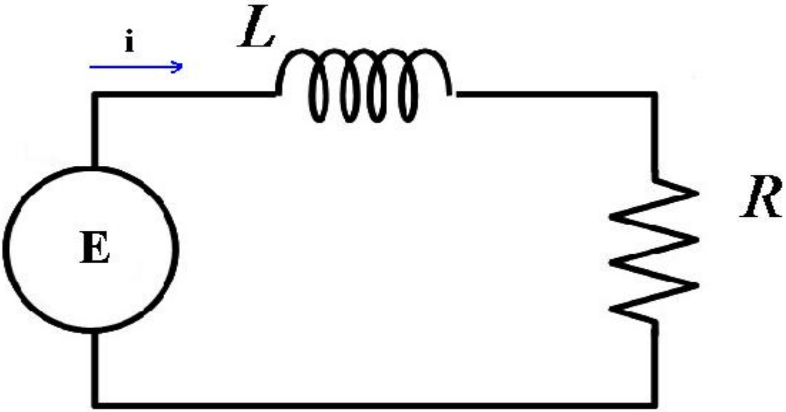


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Inductance L in henries (h), Capacitance C in farads (f), Implied voltage E in volts (V), Charge q in coulombs (C), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

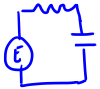
Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC



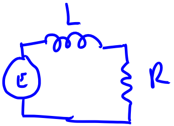
Potential Drop across = implied force

Resistor + Capacitor = E

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

1st order Linear ODE for the charge q

LR



Potential drop across ... = Implied Voltage

$$L\frac{di}{dt} + Ri = E$$

1st order linear ODE for current i.