Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a linear combination of the columns of $A$—turns out to be a vector. If the columns of $A$ are vectors in $\mathbb{R}^m$, and there are $n$ of them, then

- $A$ is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

So we can think of $A$ as an **object that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$. 
Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

**Definition:** A transformation $T$ (a.k.a. function or mapping) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Some relevant terms and notation include

- $\mathbb{R}^n$ is the **domain** and $\mathbb{R}^m$ is called the **codomain**.
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under $T$.
- The collection of all images is called the **range**.
- The notation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ may be used to indicate that $\mathbb{R}^n$ is the domain and $\mathbb{R}^m$ is the codomain.
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix $A$, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$. 
Matrix Transformation Example

Let \( A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \). Define the transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) by the mapping \( T(\mathbf{x}) = A\mathbf{x} \).

(a) Find the image of the vector \( \mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \) under \( T \).
(b) Determine a vector $\mathbf{x}$ in $\mathbb{R}^2$ whose image under $T$ is \[
\begin{bmatrix}
-4 \\
-4 \\
4
\end{bmatrix}.
\]
\[ A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \]

(c) Determine if \[ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \] is in the range of \( T \).
Linear Transformations

**Definition:** A transformation \( T \) is **linear** provided

(i) \( T(u + v) = T(u) + T(v) \) for every \( u, v \) in the domain of \( T \), and

(ii) \( T(cu) = cT(u) \) for every scalar \( c \) and vector \( u \) in the domain of \( T \).

Every matrix transformation (e.g. \( x \mapsto Ax \)) is a linear transformation. And it turns out that every linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) can be expressed in terms of matrix multiplication.
A Theorem About Linear Transformations:

If $T$ is a linear transformation, then

$$T(0) = 0,$$

$$T(cu + dv) = cT(u) + dT(v)$$

for scalars $c, d$ and vectors $u, v$.

And in fact

$$T(c_1u_1 + c_2u_2 + \cdots + c_ku_k) = c_1T(u_1) + c_2T(u_2) + \cdots + c_kT(u_k).$$
Example

Let $r$ be a nonzero scalar. The transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(x) = rx$$

is a linear transformation$^1$.
Show that $T$ is a linear transformation.

$^1$It's called a **contraction** if $0 < r < 1$ and a **dilation** when $r > 1$.
Figure: Geometry of dilation $\mathbf{x} \mapsto 2\mathbf{x}$. The 4 by 4 square maps to an 8 by 8 square.
Section 1.9: The Matrix for a Linear Transformation

**Elementary Vectors:** We’ll use the notation $e_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the $i^{th}$ position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in $\mathbb{R}^3$ they would be

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity $I_n$. 
Matrix of Linear Transformation

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) be a linear transformation, and suppose

\[
T(e_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}.
\]

Use the fact that \( T \) is linear, and the fact that for each \( \mathbf{x} \) in \( \mathbb{R}^2 \) we have

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_2 \mathbf{e}_1 + x_2 \mathbf{e}_2
\]

to find a matrix \( A \) such that

\[
T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2.
\]
\[ T(e_1) = \begin{bmatrix} 0 & 1 \\ -2 & 4 \end{bmatrix}, \quad \text{and} \quad T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix} \]
Theorem

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix $A$ such that

$$T(x) = Ax \quad \text{for every} \quad x \in \mathbb{R}^n.$$ 

Moreover, the $j^{th}$ column of the matrix $A$ is the vector $T(e_j)$, where $e_j$ is the $j^{th}$ column of the $n \times n$ identity matrix $I_n$. That is,

$$A = [T(e_1) \ T(e_2) \ \cdots \ T(e_n)].$$

The matrix $A$ is called the **standard matrix** for the linear transformation $T$. 
Example
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the scaling transformation (contraction or dilation for $r > 0$) defined by

$$T(x) = rx, \quad \text{for positive scalar } r.$$ 

Find the standard matrix for $T$. 
Example

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle $\phi$. Find the standard matrix for $T$. 
Example

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the projection transformation that projects each point onto the $x_1$ axis

$$T \left( \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right] \right) = \left[ \begin{array}{c} x_1 \\ 0 \end{array} \right].$$

Find the standard matrix for $T$.

\[2\text{See pages 73–75 in Lay for matrices associated with other geometric transformation on } \mathbb{R}^2\]
One to One, Onto

**Definition:** A mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be **onto** \( \mathbb{R}^m \) if each \( b \) in \( \mathbb{R}^m \) is the image of at least one \( x \) in \( \mathbb{R}^n \)—i.e. if the range of \( T \) is all of the codomain.

**Definition:** A mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be **one to one** if each \( b \) in \( \mathbb{R}^m \) is the image of at most one \( x \) in \( \mathbb{R}^n \).
Determine if the transformation is one to one, onto, neither or both.

\[ T(x) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} x. \]
Some Theorems

**Theorem:** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then $T$ is one to one if and only if the homogeneous equation $T(x) = 0$ has only the trivial solution.
Some Theorems

**Theorem:** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let $A$ be the standard matrix for $T$. Then

(i) $T$ is onto if and only if the columns of $A$ span $\mathbb{R}^m$, and

(ii) $T$ is one to one if and only if the columns of $A$ are linearly independent.
Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that $T$ is one to one. Is $T$ onto?