February 7 Math 1190 sec. 62 Spring 2017

Section 2.1: Rates of Change and the Derivative

We opened by saying that Calculus is concerned with the way in which quantities change. An obvious example of change is motion of an object in space (change of position).

Here we introduce the idea of *rate of change* and the mathematical formulation of this called a *derivative*.

Though we'll use **rectilinear motion** (i.e. movement along a straight line) as an illustrative example, the concept can be applied to many processes in physics, chemistry, biology, business, and the list goes on!

Motivational Example:

Suppose a ball is dropped from the top of the Space Needle 605 feet high. According to Galileo's law, the distance $s(t)$ feet the ball has fallen after t seconds is (neglecting wind drag)

$$s(t) = 16t^2.$$

The position of the ball relative to the top of the tower is changing. We can consider the ball's velocity.

We define **average velocity** as

$$\text{change in position} \quad \div \quad \text{change in time.}$$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 0$ to $t = 2$.

$s = 16t^2$ ft $\qquad t \sim$ seconds

$s(2) = 16(2^2)$ ft $= 64$ ft $\qquad 2$ sec $- 0$ sec $= 2$ sec

$s(0) = 16(0^2)$ ft $= 0$ ft

Avg. velocity $= \dfrac{64\text{ ft} - 0\text{ ft}}{2\text{ sec}} = \dfrac{64}{2}\dfrac{\text{ft}}{\text{sec}} = 32\dfrac{\text{ft}}{\text{sec}}$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 2$ to $t = 4$.

$S(2) = 64 \text{ ft}$

$S(4) = 16(4^2) \text{ ft} = 256 \text{ ft}$

$4 \sec - 2 \sec = 2 \sec$

$$\text{Avg. velocity} = \frac{256 \text{f} - 64 \text{ft}}{2 \sec} = \frac{192 \text{ ft}}{2 \sec} = 96 \frac{\text{ft}}{\sec}.$$

Here's a tougher question...

What is the *instantaneous velocity* when $t = 2$?

This is tough because we don't have 2 positions and 2 times.

We'll take two times, say $t = 2$ and $t = 2 + \Delta t$ for Δt a small time increment.

$$\text{avg velocity} = \frac{S(2+\Delta t) - S(2)}{2 + \Delta t - 2} = \frac{S(2+\Delta t) - S(2)}{\Delta t}$$

Estimating instantaneous velocity using intervals of decreasing size...

Δt	$\frac{s(2+\Delta t)-s(2)}{\Delta t}$	Δt	$\frac{s(2+\Delta t)-s(2)}{\Delta t}$
1	80	-1	48
0.1	65.6	-0.1	62.4
0.05	64.8	-0.05	63.2
0.01	64.16	-0.01	63.84

we're trying to compute

$$\lim_{\Delta t \to 0} \frac{s(2+\Delta t)-s(2)}{\Delta t}$$

looks like it's headed to 64 $\frac{ft}{sec}$

Instantaneous Velocity

If we consider the independent variable t and dependent variable $s = f(t)$, we note that the average velocity has the form

$$\frac{\text{change in } s}{\text{change in } t} = \frac{\Delta s}{\Delta t}$$

Definition: We define the instantaneous velocity v (simply called *velocity*) at the time t_0 as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

provided this limit exists.

Example

An object moves along the x-axis such that its distance s from the origin at time t is given by $s = \sqrt{2t}$. If s is in inches and t is in seconds, determine the object's velocity at $t_0 = 3$ sec.

$$v = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{here} \quad f(t) = \sqrt{2t} \quad \text{and} \quad t_0 = 3$$

$$f(t) = \sqrt{2t} \text{ in} \quad \text{and} \quad f(t_0) = f(3) = \sqrt{2\cdot 3} \text{ in} = \sqrt{6} \text{ in}$$

$$\text{So} \quad \frac{f(t) - f(3)}{t - 3} = \frac{\sqrt{2t} - \sqrt{6}}{t - 3} \ \frac{\text{in}}{\text{sec}}$$

$$v = \lim_{t \to 3} \frac{\sqrt{2t} - \sqrt{6}}{t - 3}$$

$$= \lim_{t \to 3} \left(\frac{\sqrt{2t} - \sqrt{6}}{t - 3}\right)\left(\frac{\sqrt{2t} + \sqrt{6}}{\sqrt{2t} + \sqrt{6}}\right)$$

$$= \lim_{t \to 3} \frac{2t - 6}{(t-3)(\sqrt{2t} + \sqrt{6})}$$

$$= \lim_{t \to 3} \frac{2\cancel{(t-3)}}{\cancel{(t-3)}(\sqrt{2t} + \sqrt{6})}$$

$$= \lim_{t \to 3} \frac{2}{\sqrt{2t} + \sqrt{6}}$$

$$= \frac{2}{\sqrt{2\cdot 3}+\sqrt{6}} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

The velocity @ $t_0 = 3$ sec is

$$v = \frac{1}{\sqrt{6}}\ \frac{\text{in}}{\text{sec}}$$

Question

A cannon ball is fired from the ground so that it's distance from the ground after t seconds is given by $s = 80t - 16t^2$ feet. Which of the following limits would be used to determine the ball's velocity at $t_0 = 3$ seconds?

$$v = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

(a) $\displaystyle \lim_{t\to 0} \frac{80t - 16t^2 - 96}{t}$

(b) $\displaystyle \lim_{t\to 3} \frac{80t - 16t^2 - 96}{t-3}$

(c) $\displaystyle \lim_{t\to 0} \frac{80t - 16t^2 - 96}{t-3}$

(d) $\displaystyle \lim_{t\to 3} \frac{80t - 16t^2 - 96}{t}$

Observation

Note that the average velocity has the form $\frac{\Delta s}{\Delta t}$. This ratio (should) look familiar. If we think graphically, with $s = f(t)$

$$\frac{\Delta s}{\Delta t} = \frac{\text{rise}}{\text{run}} = \text{slope}$$

We're back to the **tangent line** problem from the beginning of chapter 1!

The Tangent Line Problem

Given a graph of a function $y = f(x)$:

A **secant** line is a line connecting two points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ on the graph. The slope of a secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Recall that if $P = (c, f(c))$ and $Q = (x, f(x))$ are distinct points, we denoted the slope of the secant line

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$

We had defined the slope of the tangent line as

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{if this limit exists.}$$

Example

Find the slope of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(-1, -1)$.

For $f(x) = \frac{1}{x}$, the tangent line @ $(-1,-1)$ has slope

$$m_{tan} = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{\frac{1}{x} - (-1)}{x+1}$$

$$= \lim_{x \to -1} \frac{\frac{1}{x} + 1}{x+1}$$

$$= \lim_{x \to -1} \frac{\frac{1}{x} + \frac{x}{x}}{x+1}$$

$$= \lim_{x \to -1} \frac{\frac{1+x}{x}}{x+1}$$

$$= \lim_{x \to -1} \frac{1+x}{x} \cdot \frac{1}{x+1}$$

$$= \lim_{x \to -1} \frac{1}{x} = \frac{1}{-1} = -1$$

$$m_{tan} = -1 \quad @ \quad (-1, 1)$$

Example Continued...

Find the equation of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(-1, -1)$.

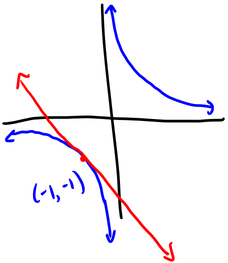
Slope $m_{tan} = -1$

$y - (-1) = -1\,(x - (-1))$

$y + 1 = -(x + 1)$

$y + 1 = -x - 1$

$\Rightarrow \boxed{y = -x - 2}$



Tangent Line

Theorem: Let $y = f(x)$ and let c be in the domain of f. If the slope m_{tan} exists at the point $(c, f(c))$, then the equation of the line tangent to the graph of f at this point is

$$y = m_{tan}(x - c) + f(c).$$