February 8 MATH 1112 sec. 54 Spring 2019

Section 5.4: Properties of Logarithms

Let's recall the basic properties of exponentials. The logarithms have analog properties.

Let $a > 0$, with $a \neq 1$. Then for any real x and y

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$

Log of a Product

Theorem: Let M and N be any positive numbers and $a > 0$ with $a \neq 1$. Then

$$\log_a(MN) = \log_a(M) + \log_a(N).$$

Illustrative Example:

$$\log_2(16) = \log_2(2 \cdot 8) = \log_2(2) + \log_2(8).$$

Note that this equation is the true statement

$$4 = 1 + 3.$$

Question

Here's the meat of our theorem: $\log_a(MN) = \log_a(M) + \log_a(N)$.

Which of the following is equivalent to $\log_3(15)$?

(a) $\log_3(5) + \log_3(3)$ $\log_3(5 \cdot 3)$

(b) $\log_3(10) + \log_3(5)$

(c) $\log_3(3) \cdot \log_3(5)$

(d) all of the above are equivalent

(e) none of the above is equivalent

Log of a Power

Theorem: Let M be any positive number, a be any positive number different from 1, and p be any real number. Then

$$\log_a(M^p) = p \log_a(M).$$

Illustrative Example:

$$\log_2(64) = \log_2(4^3) = 3 \log_2(4).$$

Note that this equation is the true statement

$$6 = 3 \cdot 2.$$

Question

Here's the meat of our theorem: $\log_a(M^p) = p\log_a(M)$.

Which of the following is equivalent to $\log_7(125)$?

(a) $5\log_7(25)$

$\log_7(5^3)$

(b) $3\log_7(5)$

(c) $(\log_7(5))^3$

(d) all of the above are equivalent

(e) none of the above is equivalent

Log of a Quotient

Theorem: Let M and N be any positive numbers and $a > 0$ with $a \neq 1$. Then

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N).$$

Illustrative Example:

$$\log_2(4) = \log_2\left(\frac{16}{4}\right) = \log_2(16) - \log_2(4).$$

Note that this equation is the true statement

$$2 = 4 - 2.$$

Question

Here's the meat of our theorem: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$.

Which of the following is equivalent to $\log_3(5)$?

(a) $\log_3(15) - \log_3(3)$ $= \log_3\left(\frac{15}{3}\right)$

(b) $\log_3(10) - \log_3(5)$

(c) $\frac{\log_3(15)}{\log_3(3)}$

(d) all of the above are equivalent

(e) none of the above is equivalent

Question

Properties of logarithms are sometimes confused with erroneous, somewhat look-alike expressions.
Which of the following is FALSE?

(a) $\ln(xy) = (\ln x)(\ln y)$

(b) $\log_2(x) = \ln(x^2)$

(c) $\log_4(2 + 7) = \log_4(2) + \log_4(7)$

(d) $(\log(10))^5 = \log(10^5)$ $\log(10^5) = 5$

$= 1$ 5

(e) All of the above are false.

Example: Using the properties together

Express the following as a sum, difference, and multiple of logarithms.

$$\log_a \sqrt{\frac{a^3 b^5}{\sqrt[3]{a}\sqrt{b+1}}}$$

We'll use

$\log_a(MN) = \log_a M + \log_a N$

$\log_a(M^p) = p \log_a M$

$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

$$= \log_a \left(\frac{a^3 b^5}{a^{1/3}(b+1)^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log_a \left(\frac{a^3 b^5}{a^{1/3}(b+1)^{1/2}}\right)$$

$$\frac{a^3}{a^{1/3}} = a^{3-\frac{1}{3}} = a^{\frac{9}{3}-\frac{1}{3}} = a^{8/3}$$

$$= \frac{1}{2}\log_a \left(\frac{a^{8/3} b^5}{(b+1)^{1/2}}\right)$$

$$= \frac{1}{2}\left(\log_a\left(a^{8/3}b^5\right) - \log_a (b+1)^{1/2}\right)$$

$$= \frac{1}{2}\left(\log_a a^{8/3} + \log_a b^5 - \log_a (b+1)^{1/2}\right)$$

$$= \frac{1}{2}\left(\frac{8}{3}\log_a a + 5\log_a b - \frac{1}{2}\log_a (b+1)\right)$$

$$= \frac{1}{2}\left(\frac{8}{3}\cdot 1 + 5\log_a b - \frac{1}{2}\log_a (b+1)\right)$$

$$= \frac{4}{3} + \frac{5}{2}\log_a b - \frac{1}{4}\log_a (b+1)$$

Question

Which of the following expressions is equivalent to

$$\log_2\left(x^3\sqrt{y^2-1}\right)$$

(a) $\log_2(x^3) - \frac{1}{2}\log_2(y^2-1)$

(b) $\frac{3}{2}\log_2(x(y^2-1))$

(c) $3\log_2(x) + \frac{1}{2}\log_2(y^2-1)$

(d) $3\log_2(x) + \frac{1}{2}\log_2(y^2) - \frac{1}{2}\log_2(1)$

Summary of Log Properties

Assume each expression is well defined.

(i) Change of base: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$

(ii) Log of Product: $\log_a(MN) = \log_a(M) + \log_a(N)$

(iii) Log of Power: $\log_a(M^p) = p\log_a(M)$

(iv) Log of Quotient: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$

(v) Inverse Function: $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$

(vi) Special Values: $\log_a(1) = 0$, $\log_a(a) = 1$, and $\log_a(0)$ is never defined.