

Section 5: First Order Equations Models and Applications

RC Series Circuit: The charge $q(t)$ on the capacitor of an RC-series circuit with resistance R , capacitance C , and implied electromotive force E is governed by

$$R \frac{dq}{dt} + \frac{1}{C}q = E(t).$$

LR Series Circuit: The current $i = \frac{dq}{dt}$ in an LR-series circuit with resistance R , inductance L , and implied electromotive force E is governed by

$$L \frac{di}{dt} + Ri = E(t).$$

Measurable Quantities & Units

Resistance R in ohms (Ω), Implied voltage E in volts (V),

Inductance L in henries (h), Charge q in coulombs (C),

Capacitance C in farads (f), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \frac{dq}{dt}$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance $5 \times 10^{-6} f$. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \rightarrow \infty$.

$$R \frac{dq}{dt} + \frac{1}{C} q = E(t)$$

$$R = 1000, \quad C = 5 \cdot 10^{-6}$$

$$E(t) = 200$$

$$1000 \frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}} q = 200$$

$$i(0) = q'(0) = \frac{2}{5}$$

Standard form

$$\frac{dq}{dt} + 200q = \frac{1}{5}$$

$$* \frac{1}{5 \cdot 10^{-6}} = \frac{10^6}{5(10^3)} = \frac{10^3}{5} = 200$$

$$\frac{200}{1000} = \frac{1}{5}$$

$$\mu = e^{\int 200 dt} = e^{200t}$$

$$\frac{d}{dt} (e^{200t} q) = \frac{1}{5} e^{200t}$$

$$e^{200t} q = \int \frac{1}{5} e^{200t} dt = \frac{1}{5(200)} e^{200t} + k$$

$$q(t) = \frac{1}{1000} + k e^{-200t}$$

$$\text{Use } q'(0) = \frac{2}{5}$$

$$q'(t) = -200 k e^{-200t}$$

$$q'(0) = -200 k e^0 = \frac{2}{5}$$

$$k = \frac{2}{5(-200)} = \frac{-1}{500}$$

The charge on the capacitor

$$q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}$$

Note $q = q_p + q_c$ where $q_p = \frac{1}{1000}$ and $q_c = \frac{-1}{500} e^{-200t}$

q_p is a steady state and

q_c is the transient state

The long time charge

$$\lim_{t \rightarrow \infty} q(t) = \lim_{t \rightarrow \infty} \left(\frac{1}{1000} - \frac{1}{500} e^{-200t} \right)$$

$$= \frac{1}{1000} - 0$$

$$= \frac{1}{1000}$$

The long time charge is $\frac{1}{1000}$ C .

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t . Find the concentration of the mixture in the tank at $t = 5$ minutes.

A salt "mass" variable in lb

t time variable in minutes

A Classic Mixing Problem

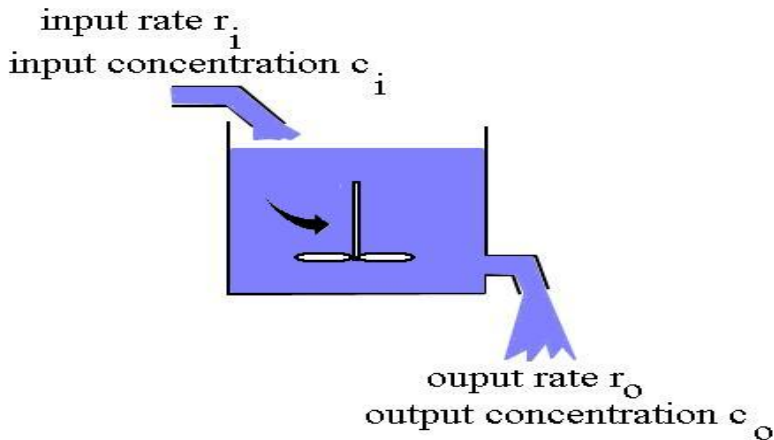


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \text{input rate} \\ \text{of salt} \end{array} \right) - \left(\begin{array}{c} \text{output rate} \\ \text{of salt} \end{array} \right)$$

r - fluid flow rate
 C - concentration
 i - in
 o - out

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} = r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} = r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}$$

Concentration
in the
tank

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}$$

↑
initial
fluid
volume

This equation is first order linear.

In general, we are given

$A(0)$ for an IVP

↑
 c_0

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t . Find the concentration of the mixture in the tank at $t = 5$ minutes.

$$r_i = 5 \frac{\text{gal}}{\text{min}}, \quad r_o = 5 \frac{\text{gal}}{\text{min}}, \quad A(0) = 0$$

$$c_i = 2 \frac{\text{lb}}{\text{gal}}, \quad V(0) = 500 \text{ gal}$$

We'll pull this together, set up and solve the

IVP next time.