February 8 Math 2306 sec. 54 Spring 2019

Section 5: First Order Equations Models and Applications

RC Series Circuit: The charge $q(t)$ on the capacitor of an RC-series circuit with resistance R, capacitance C, and implied electromotive force E is governed by

$$R\frac{dq}{dt}+\frac{1}{C}q=E(t).$$

LR Series Circuit: The current $i=\frac{dq}{dt}$ in an LR-series circuit with resistance R, inductance L, and implied electromotive force E is governed by

$$L\frac{di}{dt}+Ri=E(t).$$

Measurable Quantities & Units

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

$R = 1000 \qquad C = 5 \cdot 10^{-6}$

$E(t) = 200$

$i(0) = q'(0) = \frac{2}{5}$

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200$$

Standard form

$$* \quad \frac{\frac{1}{5 \cdot 10^{-6}}}{1000} = \frac{10^6}{5(10^3)} = \frac{10^3}{5} = 200$$

$$\frac{dq}{dt} + 200q = \frac{1}{5}$$

$$\frac{200}{1000} = \frac{1}{5}$$

$$\mu = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$e^{200t} q = \int \frac{1}{5} e^{200t}\,dt = \frac{1}{5(200)} e^{200t} + k$$

$$q(t) = \frac{1}{1000} + k e^{-200t}$$

Apply $q'(0) = \frac{2}{5}$ $\quad q'(t) = -200 k e^{-200t}$

$$q'(0) = -200 k e^{0} = \frac{2}{5} \Rightarrow k = \frac{2}{5(-200)} = \frac{-1}{500}$$

The charge q on the capacitor

$$q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}$$

Note that $q = q_p + q_c$ where

$$q_p = \frac{1}{1000}$$ q_p is the steady state

$$q_c = \frac{-1}{500} e^{-200t}$$ q_c is the transient state

The long time charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \left(\frac{1}{1000} - \frac{1}{500} e^{-200t} \right)$$

$$= \frac{1}{1000} - 0$$

$$= \frac{1}{1000}$$

The long time charge is $\frac{1}{1000}$ C.

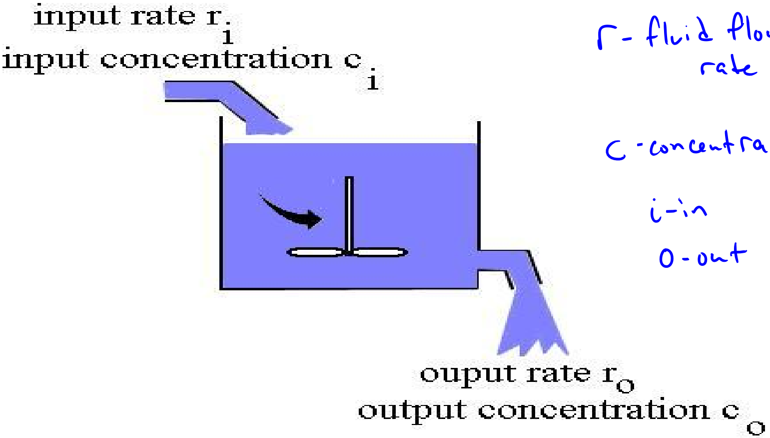
A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A - salt "mass" variable in lbs

t - time variable in minutes

A Classic Mixing Problem



r - fluid flow rate

c - concentration

i - in

o - out

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array} \right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array} \right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

concentration in the tank

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

initial volume

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

c_o

This equation is first order linear.

Usually $A(0)$ is given for an IVP

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$r_i = 5\,\frac{\text{gal}}{\text{min}}$ $\quad r_o = 5\,\frac{\text{gal}}{\text{min}}$ $\quad A(0) = 0$ (pure water)

$c_i = 2\,\frac{\text{lb}}{\text{gal}}$ $\quad V(0) = 500$ gal

$$c_o = \frac{A}{V} = \frac{A \;\; \text{lb}}{500\text{ gal} + (5\text{ gal/min} - 5\text{ gal/min})\,t} = \frac{A}{500}\,\frac{\text{lb}}{\text{gal}}$$

We will pull this together to write the IVP and solve it next time.