Section 5: First Order Equations Models and Applications

A Nonlinear Modeling Problem: A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity $M$ of the environment and the current population. Determine the differential equation satisfied by $P$.

The rate of change of the population is $\frac{dP}{dt}$. We’re told it is jointly proportional to $P$ and $M - P$ (difference between carrying capacity $M$ and $P$).

Hence

$$\frac{dP}{dt} = kP(M - P)$$

for some constant $k$.

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1The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.
Logistic Differential Equation

The equation
\[ \frac{dP}{dt} = kP(M - P), \quad k, M > 0 \]
is called a logistic growth equation.

Solve this equation\(^2\) and show that for any \( P(0) \neq 0, P \to M \) as \( t \to \infty \).

Eqn is separable
\[
\frac{1}{P(M - P)} \frac{dP}{dt} = k \quad \Rightarrow \quad \int \frac{1}{P(M - P)} \, dp = \int k \, dt
\]

\(^2\)The partial fraction decomposition
\[
\frac{1}{P(M - P)} = \frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right)
\]
is useful.
\[
\int \frac{1}{m} \left( \frac{1}{p} + \frac{1}{m-p} \right) \, dp = \int k \, dt
\]

\[
\int \left( \frac{1}{p} + \frac{1}{m-p} \right) \, dp = \int km \, dt
\]

\[\ln |p| - \ln |m-p| = km t + C\]

\[\ln \left| \frac{p}{m-p} \right| = km t + C\]

Exponentiate:

\[\left| \frac{p}{m-p} \right| = e^{km t + C} = c e^{km t}\]

Letting \(A = e^C\) or \(-e^C\) or 0.
\[
\frac{\rho}{m - \rho} = Ae^{kmt}
\]

Applying \( P(0) = P_0 \) \( \Rightarrow \frac{P_0}{m - P_0} = Ae^0 = A \)

\[
A = \frac{P_0}{m - P_0}
\]

we'll come back to this.

\[
P = Ae^{kmt} (m - P) = AMe^{kmt} - APe^{kmt}
\]

\[
P + APe^{kmt} = AMe^{kmt}
\]

\[
(1 + Ae^{kmt}) P = AMe^{kmt}
\]
\[ P = \frac{Ae^{kt}}{1 + Ae^{kt}} \]

Using \[ A = \frac{P_0}{M - P_0} \]

\[ P = \frac{P_0}{M - P_0} \frac{Me^{kt}}{1 + \frac{P_0}{M - P_0}e^{kt}} \]

\[ \left( \frac{M - P_0}{M - P_0} \right) \] Clear fractions

Finally, \[ P(t) = \frac{P_0Me^{kt}}{M - P_0 + P_0e^{kt}} \]

The solution to the W.P.
Looking at the long time population

\[
\lim_{t \to \infty} p(t) = \lim_{t \to \infty} \frac{p_0 M e^{k M t}}{m - p_0 + p_0 e^{k M t}} = \frac{\infty}{\infty}
\]

apply l'Hôpital's rule

\[
= \lim_{t \to \infty} \frac{p_0 M (k M e^{k M t})}{p_0 (k M e^{k M t})}
\]

\[
= \lim_{t \to \infty} M = M
\]

So \( p(t) \to M \) as \( t \to \infty \) for any \( p_0 \neq 0 \).
Recall that an \( n^{th} \) order linear IVP consists of an equation

\[
\begin{align*}
    a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y &= g(x) \\
\end{align*}
\]

to solve subject to conditions

\[
    y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.
\]

The problem is called **homogeneous** if \( g(x) \equiv 0 \). Otherwise it is called **nonhomogeneous**.
Theorem: Existence & Uniqueness

**Theorem:** If $a_0, \ldots, a_n$ and $g$ are continuous on an interval $I$, $a_n(x) \neq 0$ for each $x$ in $I$, and $x_0$ is any point in $I$, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on $I$.

Put differently, we’re guaranteed to have a solution exist, and it is the only one there is!
Homogeneous Equations

We’ll consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

and assume that each \(a_i\) is continuous and \(a_n\) is never zero on the interval of interest.

**Theorem:** If \(y_1, y_2, \ldots, y_k\) are all solutions of this homogeneous equation on an interval \(I\), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \(I\) for any choice of constants \(c_1, \ldots, c_k\).

This is called the **principle of superposition**.
Corollaries

(i) If $y_1$ solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any nontrivial solution(s), and
- since $y_1$ and $cy_1$ aren’t truly different solutions, what criteria will be used to call solutions distinct?
Linear Dependence

Definition: A set of functions \( f_1(x), f_2(x), \ldots, f_n(x) \) are said to be linearly dependent on an interval \( I \) if there exists a set of constants \( c_1, c_2, \ldots, c_n \) with at least one of them being nonzero such that

\[
c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.
\]

A set of functions that is not linearly dependent on \( I \) is said to be linearly independent on \( I \).
Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

We want to show that there exists numbers $c_1, c_2, c_3$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all real} \ x$$

Recall $\sin^2 x + \cos^2 x = 1$

Taking $c_1 = c_2 = 1$ and $c_3 = -1 \ (\text{not all zero})$

$$1 f_1(x) + 1 f_2(x) - 1 f_3(x) = \sin^2 x + \cos^2 x - 1 = 0$$
Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We need to show that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real $x$ only if $c_1 = 0$ and $c_2 = 0$.

Suppose $c_1 \sin x + c_2 \cos x = 0$ for all real $x$.

It must be true when $x = 0$,

$\begin{align*}
c_1 \sin 0 + c_2 \cos 0 &= 0 \\
c_1 \cdot 0 + c_2 \cdot 1 &= 0 \quad \Rightarrow \quad c_2 = 0
\end{align*}$
Since \( c_2 = 0 \), the equation is

\[
    c_1 \sin x = 0 \quad \text{for all real } x.
\]

This holds when \( x = \frac{\pi}{2} \), so

\[
    c_1 \sin \frac{\pi}{2} = 0
    \]

\[
    c_1 \cdot 1 = 0 \quad \Rightarrow \quad c_1 = 0
    \]

Both \( c_1 \) and \( c_2 \) must be zero.
Determine if the set is Linearly Dependent or Independent on \((-\infty, \infty)\)

\[
f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2
\]

Consider \(c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0\) for all \(x\)

\[
c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0
\]

Let's collect like terms

\[
(c_1 - c_3) x^2 + (4c_2 + c_3) x = 0
\]

We can make these coefficients zero by
Taking \( c_1 = c_3 \) and \( c_2 = \frac{1}{4} c_3 \)

A nonzero set of \( c \)'s is \( c_1 = 4, c_2 = -1, c_3 = 4 \)

\[ 4f_1(x) - f_2(x) + 4f_3(x) = 0 \]

\[ 4x^2 - 4x + 4(x - x^2) = 0 \quad \text{for all } x. \]

The set is linearly dependent,