February 8 Math 2306 sec. 60 Spring 2018

Section 5: First Order Equations Models and Applications

A Nonlinear Modeling Problem: A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

The rate of change of the population is $\frac{dP}{dt}$. We're told it is jointly proportional to

P and $M - P$ (difference between carrying capacity M and P).

Hence
$$\frac{dP}{dt} = kP(M - P) \quad \text{for some constant } k.$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

The ODE is separable $\qquad P(0) = P_0$

$$\frac{1}{P(M-P)} \frac{dP}{dt} = k \quad \Rightarrow \quad \int \frac{1}{P(M-P)}\, dP = \int k\, dt$$

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\, dt$$

[2] The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP=\int kM\,dt$$

$$\ln|P|-\ln|M-P|=kMt+C$$

$$\ln\left|\frac{P}{M-P}\right|=kMt+C$$

$$\left|\frac{P}{M-P}\right|=e^{kMt+C}=e^{C}e^{kMt}$$

Letting $A=e^{C}$, $-e^{C}$, or zero

$$\frac{P}{M-P}=Ae^{kMt}$$

Applying $P(0)=P_0$ $\quad \frac{P_0}{M-P_0} = Ae^0 = A$

$$\boxed{A = \frac{P_0}{M-P_0}}$$

We'll come back for this.

$$\frac{P}{M-P} = Ae^{kMt} \implies P = Ae^{kMt}(M-P)$$

$$P = AMe^{kMt} - APe^{kMt}$$

$$P + APe^{kMt} = AMe^{kMt}$$

$$(1 + Ae^{kMt})P = AMe^{kMt}$$

$$P = \frac{AMe^{kMt}}{1 + Ae^{kMt}} \qquad \text{use } A = \frac{P_0}{M - P_0}$$

$$P = \frac{\frac{P_0}{M-P_0} M e^{kMt}}{1 + \frac{P_0}{M-P_0} e^{kMt}} \left(\frac{M-P_0}{M-P_0}\right) \quad \text{clear fractions}$$

$$\boxed{P = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}}$$

Solution to the IVP.

The long time population

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} = \text{"}\frac{\infty}{\infty}\text{"}$$

use l'Hopital's rule

$$= \lim_{t\to\infty} \frac{\cancel{P_0} M (\cancel{kM} \cancel{e^{kMt}})}{\cancel{P_0}(\cancel{kM} \cancel{e^{kMt}})}$$

$$= \lim_{t\to\infty} M \quad = M$$

Hence $P(t) \to M$ as $t \to \infty$ for any $P_0 \neq 0$.

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

to solve subject to conditions

$$y(x_0)=y_0, \quad y'(x_0)=y_1, \quad \ldots, \quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

We want to show that there exists c_1, c_2, c_3 not all zero such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x.

The eqn is $c_1 \sin^2 x + c_2 \cos^2 x + c_3 = 0$

Recall $\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x + \cos^2 x - 1 = 0$ for all real x

We can take $c_1 = c_2 = 1$ and $c_3 = -1$.

Since at least one of these is nonzero, the functions are linearly dependent.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Consider $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x.

$c_1 \sin x + c_2 \cos x = 0$ for all real x.

If this holds for all x, it holds when $x = 0$.

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

$$c_1 \cdot 0 + c_2 \cdot 1 = 0 \implies c_2 = 0$$

So the equation is

$$c_1 \sin x = 0 \quad \text{for all real } x$$

This has to hold when $x = \pi/2$.

So

$$c_1 \sin\frac{\pi}{2} = 0$$

$$c_1 \cdot 1 = 0 \implies c_1 = 0.$$

So $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x

only if $c_1 = c_2 = 0$.

Hence the functions are linearly independent.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Consider $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all real x.

$$c_1 x^2 + c_2(4x) + c_3(x - x^2) = 0$$

Collect like terms

$$(c_1 - c_3)x^2 + (4c_2 + c_3)x = 0$$

Everything cancels if $c_1 = c_3$ and $c_2 = \frac{-1}{4} c_3$

A choice is $c_1 = 4$, $c_3 = 4$, $c_2 = -1$

Then $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x

$$4x^2 + (-1)(4x) + 4(x - x^2) = 0$$

They're linearly dependent.