Section 5: First Order Equations Models and Applications

A Nonlinear Modeling Problem: A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity $M$ of the environment and the current population. Determine the differential equation satisfied by $P$.

The rate of change of the population is $\frac{dP}{dt}$. We’re told it is jointly proportional to $P$ and $M - P$ (difference between carrying capacity $M$ and $P$).

Hence

$$\frac{dP}{dt} = kP(M - P)$$

for some constant $k$.

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1The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.
Logistic Differential Equation

The equation
\[ \frac{dP}{dt} = kP(M - P), \quad k, M > 0 \]
is called a logistic growth equation.

Solve this equation\(^2\) and show that for any \(P(0) \neq 0\), \(P \to M\) as \(t \to \infty\).

The ODE is separable\(^2\)

\[ \frac{1}{P(M - P)} \frac{dP}{dt} = k \quad \Rightarrow \quad \int \frac{1}{P(M - P)} \, dP = \int k \, dt \]

\[ \int \frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right) \, dP = \int k \, dt \]

\(^2\)The partial fraction decomposition

\[ \frac{1}{P(M - P)} = \frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right) \]
is useful.
\[
\int \left( \frac{1}{p} + \frac{1}{m-p} \right) dp = \int kM \, dt
\]

\[
\ln |P| - \ln |N-P| = kmt + C
\]

\[
\ln \left| \frac{p}{m-p} \right| = kmt + C
\]

\[
\left| \frac{p}{m-p} \right| = e^{kmt+C} = e^c e^{kmt}
\]

Letting \( A = e^c \), \(-e^c\), or zero

\[
\frac{p}{m-p} = A e^{kmt}
\]
Applying $P(0) = P_0$,

$$\frac{P_0}{m - p_0} = A e^0 = A$$

$$A = \frac{P_0}{m - p_0}$$

We'll come back for this.

$$\frac{P}{m - p} = A e^{kmt} \Rightarrow P = A e^{kmt} (m - p)$$

$$P = A m e^{kmt} - A P e^{kmt}$$

$$P + A P e^{kmt} = A m e^{kmt}$$

$$(1 + A e^{kmt}) P = A m e^{kmt}$$
\[ P = \frac{A M e^{k_m t}}{1 + A e^{k_m t}} \]

use \[ A = \frac{P_0}{M - P_0} \]

\[ P = \frac{P_0 M e^{k_m t}}{M - P_0}\left(1 + \frac{P_0}{M - P_0} e^{k_m t}\right) \]

Clear fractions

Solution to the IVP.
The long time population \( \lim_{t \to \infty} p(t) = \lim_{t \to \infty} \frac{p_0 M e^{k M t}}{M - p_0 + p_0 e^{k M t}} = \frac{8}{8} \)

Use l'Hopital's rule

\[ \lim_{t \to \infty} \frac{p_0 M (k M e^{k M t})}{p_0 (k M e^{k M t})} \]

\[ = \lim_{t \to \infty} M = M \]

Hence \( p(t) \to M \) as \( t \to \infty \) for any \( p_0 \neq 0 \).
Recall that an $n^{th}$ order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$ 

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.
Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and $g$ are continuous on an interval $I$, $a_n(x) \neq 0$ for each $x$ in $I$, and $x_0$ is any point in $I$, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on $I$.

Put differently, we’re guaranteed to have a solution exist, and it is the only one there is!
Homogeneous Equations

We’ll consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

and assume that each \( a_i \) is continuous and \( a_n \) is never zero on the interval of interest.

**Theorem:** If \( y_1, y_2, \ldots, y_k \) are all solutions of this homogeneous equation on an interval \( I \), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \( I \) for any choice of constants \( c_1, \ldots, c_k \).

This is called the **principle of superposition**.
Corollaries

(i) If $y_1$ solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any nontrivial solution(s), and
- since $y_1$ and $cy_1$ aren’t truly different solutions, what criteria will be used to call solutions distinct?
Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be linearly dependent on an interval $I$ if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x$ in $I$.

A set of functions that is not linearly dependent on $I$ is said to be linearly independent on $I$. 
Example: A linearly Dependent Set

The functions \( f_1(x) = \sin^2 x \), \( f_2(x) = \cos^2 x \), and \( f_3(x) = 1 \) are linearly dependent on \( I = (-\infty, \infty) \).

We want to show that there exists \( c_1, c_2, c_3 \) not all zero such that \( c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \) for all \( x \).

The condition is
\[
c_1 \sin^2 x + c_2 \cos^2 x + c_3 = 0
\]

Recall \( \sin^2 x + \cos^2 x = 1 \) \( \Rightarrow \sin^2 x + \cos^2 x - 1 = 0 \) for all \( x \).

We can take \( c_1 = c_2 = 1 \) and \( c_3 = -1 \).

Since at least one of these is nonzero, the functions are linearly dependent.
Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Consider $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real $x$.

$c_1 \sin x + c_2 \cos x = 0$ for all real $x$.

If this holds for all $x$, it holds when $x = 0$.

$c_1 \sin 0 + c_2 \cos 0 = 0$

$c_1 \cdot 0 + c_2 \cdot 1 = 0 \quad \Rightarrow c_2 = 0$
So the equation is
\[ c_1 \sin x = 0 \quad \text{for all real } x \]

This has to hold when \( x = \frac{\pi}{2} \).

So
\[ c_1 \sin \frac{\pi}{2} = 0 \]
\[ c_1 \cdot 1 = 0 \quad \Rightarrow \quad c_1 = 0. \]

So \( c_1 f_1(x) + c_2 f_2(x) = 0 \) for all real \( x \)
only if \( c_1 = c_2 = 0 \).

Hence the functions are linearly independent.
Determine if the set is Linearly Dependent or Independent on \((-\infty, \infty)\)

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]

Consider \( c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \) for all real \( x \).

\[ c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0 \]

Collect like terms

\[ (c_1 - c_3) x^2 + (4c_2 + c_3) x = 0 \]

Everything cancels if \( c_1 = c_3 \) and \( c_2 = -\frac{1}{4} c_3 \)
A choice is \( c_1 = y_1, \ c_2 = y_2, \ c_3 = -1 \)

Then \( c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \) for all \( x \)

\[ 4x^2 + (-1)(4x) + 4(x-x^2) = 0 \]

They're linearly dependent.