February 8 Math 2306 sec. 60 Spring 2019

Section 5: First Order Equations Models and Applications

RC Series Circuit: The charge $q(t)$ on the capacitor of an RC-series circuit with resistance R, capacitance C, and implied electromotive force E is governed by

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

LR Series Circuit: The current $i = \frac{dq}{dt}$ in an LR-series circuit with resistance R, inductance L, and implied electromotive force E is governed by

$$L\frac{di}{dt} + Ri = E(t).$$

Measurable Quantities & Units

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

$R = 1000 \qquad C = 5 \cdot 10^{-6}$

$E(t) = 200$

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200$$

$i(0) = q'(0) = 0.4 = \frac{2}{5}$

Standard form

$$* \quad \frac{\frac{1}{5 \cdot 10^{-6}}}{1000} = \frac{10^6}{5(1000)} = \frac{10^3}{5} = 200$$

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \quad q'(0) = \frac{2}{5}$$

$$\frac{200}{1000} = \frac{1}{5}$$

$P(t) = 200$ so $\mu(t) = e^{\int 200\,dt} = e^{200t}$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$e^{200t} q = \int \frac{1}{5} e^{200t}\,dt = \frac{1}{5(200)} e^{200t} + k$$

$$q = \frac{1}{1000} + k e^{-200t}$$

Apply the condition $q'(0) = \frac{2}{5}$

$$q'(t) = -200k\, e^{-200t}$$

$$q'(0) = -200k e^0 = \frac{2}{5} \implies k = \frac{2}{5(-200)} = \frac{-1}{500}$$

The charge on the capacitor

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}}$$

Here, $q = q_p + q_c$ where

$$q_p(t) = \frac{1}{1000} \quad \text{and} \quad q_c(t) = \frac{-1}{500} e^{-200t}$$

q_p is the steady state and q_c is the transient state

The long time charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500}e^{-200t}\right)$$

$$= \frac{1}{1000} - 0$$

$e^{-200t} \to 0$

as

$t \to \infty$

$$= \frac{1}{1000}$$

The charge approaches $\frac{1}{1000}$ C

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A salt variable in lbs

t time variable in minutes

A Classic Mixing Problem

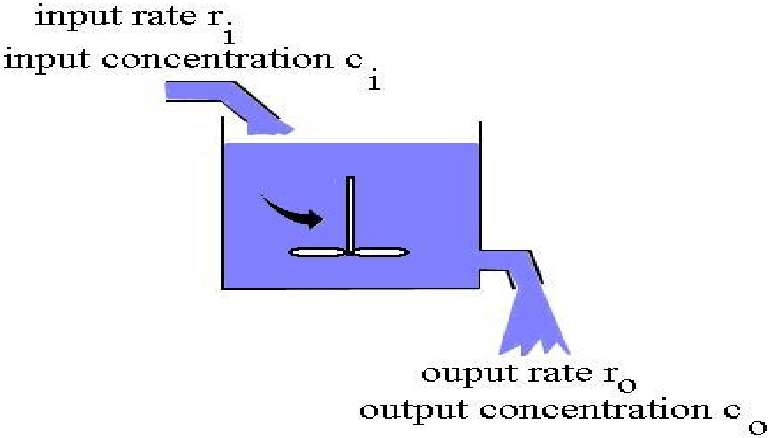

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c}\text{input rate}\\ \text{of salt}\end{array}\right) - \left(\begin{array}{c}\text{output rate}\\ \text{of salt}\end{array}\right)$$

r - fluid rate

c - concentration

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

i - in

o - out

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

the tank concentration

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

initial tank volume

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

c_o

This equation is first order linear.

Combined with $A(0)$, we get an IVP.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$r_i = 5 \frac{\text{gal}}{\text{min}}$ $\quad$ $r_o = 5 \frac{\text{gal}}{\text{min}}$

$c_i = 2 \frac{\text{lb}}{\text{gal}}$ $\quad$ $c_o = \frac{A}{V}$

$$V(t) = V(0) + (r_i - r_o)t = 500 \text{ gal} + \left(5 \frac{\text{gal}}{\text{min}} - 5 \frac{\text{gal}}{\text{min}}\right) t$$

$$= 500 \text{ gal}$$

$$c_o = \frac{A}{500} \ \frac{\text{lb}}{\text{gal}}$$

pure water $\Rightarrow$ $A(0) = 0$

We have all of the pieces for our IVP.

We'll set it up and solve it next time.