February 9 Math 1190 sec. 62 Spring 2017

Inspired by the exam, let's start off with a few questions...

Question

True or False Suppose we determine that $\lim_{x\to 1} f(x) = 5$. We can state this conclusion as

$$\lim_{x\to 1} = 5.$$

This is like saying $x=16$ so "$\sqrt{\ } = 4$".

Question

True or False: $$\frac{\sin x}{x} = 1$$

It is true that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Note: If " $\frac{\sin x}{x} = 1$ " was true, it would mean that $\sin x = x$. But we know the graphs of $y = \sin x$ and $y = x$ are radically different.

Question

$$\text{Evaluate} \quad \lim_{x\to\infty} \frac{4x^3+2x^2+3x+1}{3x^4+7x^2-x+4} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

(a) $\frac{4}{3}$

(b) ∞

(c) 0

(d) All real number since x can be anything

$$= \lim_{x\to\infty} \frac{\frac{4}{x}+\frac{2}{x^2}+\frac{3}{x^4}+\frac{1}{x^4}}{3+\frac{7}{x^2}-\frac{1}{x^3}+\frac{4}{x^4}}$$

$$= \frac{0+0+0+0}{3+0-0+0} = \frac{0}{3} = 0$$

Question

The limit $\lim_{x\to 0} \dfrac{\sin(4x)}{x} = 4$ because

we know that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

(a) $\dfrac{\sin(4x)}{x} = \dfrac{4\sin(x)}{x}$

$\lim_{x\to 0} \dfrac{\sin(4x)}{x} = \lim_{x\to 0} \dfrac{\sin(4x)}{x} \cdot \dfrac{4}{4}$ ← mult by 1

(b) $\dfrac{\sin(4x)}{x} = \sin(4) = 4$

$= \lim_{x\to 0} 4\left(\dfrac{\sin(4x)}{4x}\right)$

now, let $\theta = 4x$

if $x \to 0$, $4x \to 0$

(c) $\dfrac{\sin(4x)}{x} = \dfrac{4\sin(4x)}{4x}$

$= \lim_{\theta\to 0} 4\left(\dfrac{\sin(\theta)}{\theta}\right) = 4 \cdot 1$

Question

The population P of trout (in thousands of fish) in a certain lake at time t (in years) is given by

$$P(t) = \frac{50}{5e^{-t} + 5}.$$

The long term expected trout population $\lim_{t\to\infty} P(t)$ is

Remember $\lim_{x\to\infty} e^{-x} = 0$

i.e. $\lim_{x\to -\infty} e^{x} = 0$

(a) 50 thousand fish

(b) 10 thousand fish

(c) infinitely many fish

(d) zero fish

So $\lim_{t\to\infty} \frac{50}{5e^{-t}+5} = \frac{50}{0+5} = 10$

(Back to) Section 2.1: Rates of Change and the Derivative

We saw the same limit appear in two contexts:

(1) If $f(t)$ is the position of an object in rectilinear motion, then the instantaneous velocity of the object at time t_0 is

$$v = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{(if the limit exists.)}$$

(2) At the point $(c, f(c))$ on the graph $y = f(x)$, the slope of the line tangent to curve is

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{(if the limit exists.)}$$

Rate of Change: The Derivative

Let $y = f(x)$. For $x \neq c$ we'll call $\frac{f(x)-f(c)}{x-c}$ the average rate of change of f on the interval from x to c.

We'll call

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{the rate of change of } f \text{ at } c$$

if this limit exists.

The Derivative at a Point

Definition: Let $y = f(x)$ and let c be in the domain of f. The **derivative** of f at c is denoted $f'(c)$ and is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

The Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In addition to *the derivative of f at c*, the notation $f'(c)$ is read as

- f prime of c, or
- f prime at c.

At this point, we have several interpretations of this same **number** $f'(c)$.

- as a velocity if f is the position of a moving object,
- as a rate of change of the function f when $x = c$,
- as the slope of the line tangent to the graph of f at $(c, f(c))$.

Question

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Determine $f'(c)$ if $f(x) = 2x - x^2$ and $c = 1$.

$f(1) = 2 \cdot 1 - 1^2 = 2 - 1 = 1$

$$f'(1) = \lim_{x \to 1} \frac{2x - x^2 - 1}{x - 1}$$

compute the limit by factoring

(a) $f'(1) = 2 - 2x$

(b) $f'(1) = 2$

(c) $f'(1)$ DNE

(d) $f'(1) = 0$

Section 2.2: The Derivative as a Function

If $f(x)$ is a function, then the set of numbers $f'(c)$ for various values of c can define a new function. To proceed, we consider an alternative formulation for $f'(c)$.

If it exists, then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. Let $x = c+h$.

Then $h = x - c$. $\lim_{x \to c} h = \lim_{x \to c} (x-c) = c - c = 0$

Thus

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{c+h-c}$$

$$= \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

The Derivative Function

Let f be a function. Define the new function f' by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

called the **derivative** of f. The domain of this new function is the set

$$\{x | x \text{ is in the domain of } f, \text{ and } f'(x) \text{ exists}\}.$$

f' is read as "f prime."

Example

Let $f(x) = \sqrt{x-1}$. Identify the domain of f. Find f' and identify its domain.

Domain of f: we require $x - 1 \geq 0 \Rightarrow x \geq 1$

In interval notation, the domain is $[1, \infty)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}\right) \cdot \left(\frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}\right)$$

$$= \lim_{h\to 0} \frac{x+h-1 - (x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{1} - \cancel{x} + \cancel{1}}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x-1}} = \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

So $$\boxed{f'(x) = \frac{1}{2\sqrt{x-1}}}$$

For the domain of f' we require $x - 1 > 0$

i.e. $x > 1$. In interval notation this

is $(1, \infty)$.

Example

Find the equation of the line tangent to the graph of $f(x) = \sqrt{x-1}$ at the point $(5, 2)$.

Recall $m_{tan} = f'(5)$ we know that $f'(x) = \frac{1}{2\sqrt{x-1}}$.

So $m_{tan} = f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

The tangent line is $y - 2 = \frac{1}{4}(x-5)$

$$y = \frac{1}{4}x - \frac{5}{4} + 2 = \frac{1}{4}x - \frac{5}{4} = \frac{1}{4}x + \frac{3}{4}$$

$$\boxed{y = \frac{1}{4}x + \frac{3}{4}}$$

Example

Is there any point on the graph of $f(x) = \sqrt{x-1}$ at which the tangent line is parallel to the line $8y = x$?

$8y = x \Rightarrow y = \frac{1}{8}x$ whose slope is $\frac{1}{8}$. So the question is: Is there a point $(c, f(c))$ where

$m_{tan} = \frac{1}{8}$? $\quad m_{tan} = f'(c) = \frac{1}{2\sqrt{c-1}}$

Solve

$$\frac{1}{8} = \frac{1}{2\sqrt{c-1}}$$

(take reciprocals)

$$8 = 2\sqrt{c-1}$$

$$\Rightarrow \quad \frac{8}{2} = \sqrt{c-1} \quad \Rightarrow \quad 4 = \sqrt{c-1}$$

$$16 = c-1 \quad \Rightarrow \quad c = 17$$

The x-value of the point is 17, the y-value is

$$f(17) = \sqrt{17-1} = \sqrt{16} = 4.$$

So there is one such point, $(17,4)$.

Question

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = 2x^2$; determine $f'(x)$.

(a) $f'(x) = 4$

(b) $f'(x) = 2x$

(c) $f'(x) = \frac{2(x+h)^2 - 2x^2}{h}$

(d) $f'(x) = 4x$

Question

$f'(x) = 4x$

Let $f(x) = 2x^2$. Find the equation of the line tangent to the graph of f at the point $(2, f(2))$.

$m_{tan} = f'(2) = 4 \cdot 2 = 8$

$f(2) = 2(2^2) = 8$

(a) $y = 8x - 8$

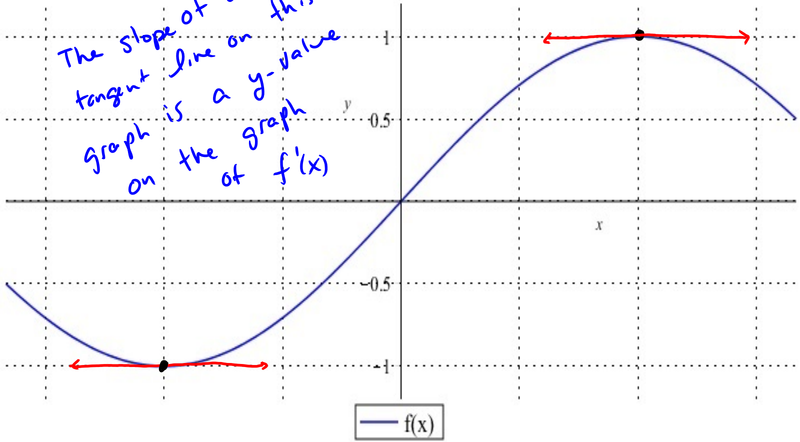
(b) $y = 4x(x-2) + 8$

(c) $y = 8x - 16$

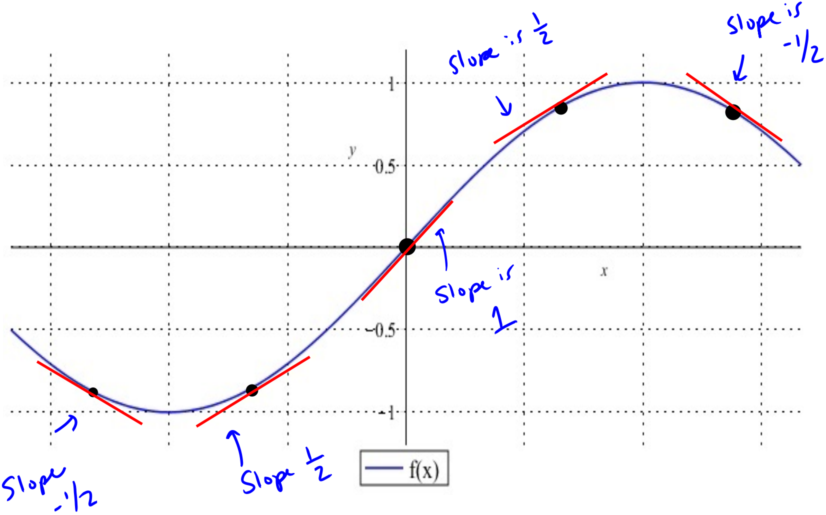
(d) $y = 4x(x-2)$

How are the functions $f(x)$ and $f'(x)$ related?

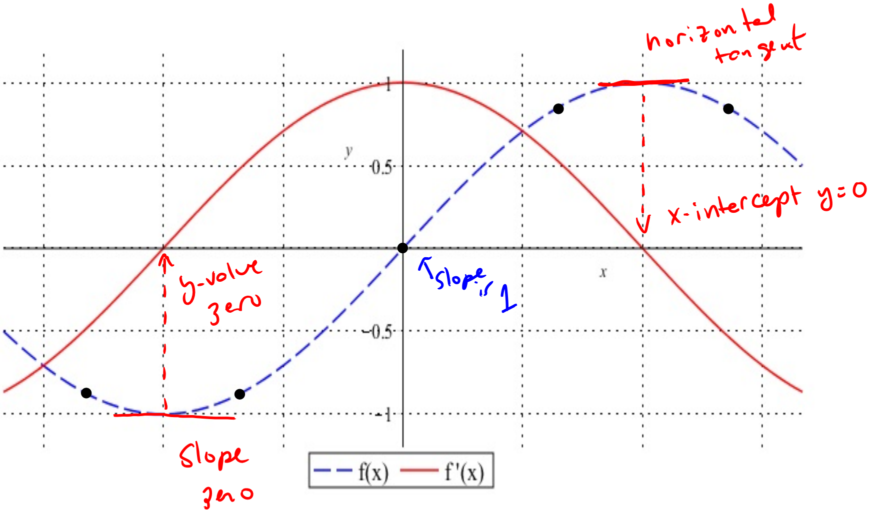
The slope of a tangent line on this graph is a y-value on the graph of $f'(x)$



How are the functions $f(x)$ and $f'(x)$ related?



How are the functions $f(x)$ and $f'(x)$ related?



Remarks:

- if $f(x)$ is a function of x, then $f'(x)$ is a new function of x (called the derivative of f)
- The number $f'(c)$ (if it exists) is the slope of the curve of $y = f(x)$ at the point $(c, f(c))$
- this is also the slope of the tangent line to the curve of y at $(c, f(c))$
- "slope of the curve", "slope of the tangent line", and "rate of change" are the same concept

Definition: A function f is said to be *differentiable* at c if $f'(c)$ exists. It is called *differentiable* on an open interval I if it is differentiable at each point in I.