# February 9 Math 1190 sec. 63 Spring 2017

Inspired by the exam, let's start off with a few questions...

# Question

**True or False** Suppose we determine that $\lim_{x\to 1} f(x) = 5$. We can state this conclusion as

$$\lim_{x\to 1} = 5.$$

This is akin to saying if $x = 16$ then

$$\sqrt{\phantom{x}} = 4.$$

# Question

**True or False:** $\quad \dfrac{\sin x}{x} = 1$

It is true that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$.

If $\dfrac{\sin x}{x} = 1$ were true, it would imply that $\sin x = x$. But the plots $y = x$ and $y = \sin x$ are clearly different.

# Question

$$\text{Evaluate} \quad \lim_{x\to\infty} \frac{4x^3+2x^2+3x+1}{3x^4+7x^2-x+4} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

(a) $\frac{4}{3}$

(b) $\infty$

(c) 0

(d) All real number since $x$ can be anything

# Question

$$\text{The limit} \quad \lim_{x\to 0} \frac{\sin(4x)}{x} = 4 \quad \text{because}$$

(a) $\dfrac{\sin(4x)}{x} = \dfrac{4\sin(x)}{x}$

(b) $\dfrac{\sin(4x)}{x} = \sin(4) = 4$

(c) $\dfrac{\sin(4x)}{x} = \dfrac{4\sin(4x)}{4x}$

# Question

The population $P$ of trout (in thousands of fish) in a certain lake at time $t$ (in years) is given by

$$P(t) = \frac{50}{5e^{-t} + 5}.$$

The long term expected trout population $\lim_{t\to\infty} P(t)$ is

$$\lim_{x\to -\infty} e^{x} = 0$$

(a) 50 thousand fish

(b) 10 thousand fish

$$\lim_{t\to\infty} \frac{50}{5e^{-t}+5} = \frac{50}{5\cdot 0+5}$$

50

(c) infinitely many fish

$$= \frac{50}{5} = 10$$

(d) zero fish

# (Back to) Section 2.1: Rates of Change and the Derivative

We saw the same limit appear in two contexts:

(1) If $f(t)$ is the position of an object in rectilinear motion, then the instantaneous velocity of the object at time $t_0$ is

$$v = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{(if the limit exists.)}$$

(2) At the point $(c, f(c))$ on the graph $y = f(x)$, the slope of the line tangent to curve is

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{(if the limit exists.)}$$

# Rate of Change: The Derivative

Let $y = f(x)$. For $x \neq c$ we'll call $\frac{f(x)-f(c)}{x-c}$ the average rate of change of $f$ on the interval from $x$ to $c$.

We'll call

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{the rate of change of } f \text{ at } c$$

if this limit exists.

# The Derivative at a Point

**Definition:** Let $y = f(x)$ and let $c$ be in the domain of $f$. The **derivative** of $f$ at $c$ is denoted $f'(c)$ and is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

# The Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In addition to *the derivative of $f$ at $c$*, the notation $f'(c)$ is read as

- $f$ prime of $c$, or
- $f$ prime at $c$.

At this point, we have several interpretations of this same **number** $f'(c)$.

- as a velocity if $f$ is the position of a moving object,
- as a rate of change of the function $f$ when $x = c$,
- as the slope of the line tangent to the graph of $f$ at $(c, f(c))$.

# Question

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Determine $f'(c)$ if $f(x) = 2x - x^2$ and $c = 1$.

(a) $f'(1) = 2 - 2x$

(b) $f'(1) = 2$

(c) $f'(1)$ DNE

(d) $f'(1) = 0$

$$f(1) = 2 \cdot 1 - 1^2 = 2 - 1 = 1$$

$$f'(1) = \lim_{x \to 1} \frac{2x - x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{-(x^2 - 2x + 1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{-(x-1)^2}{x-1} = \lim_{x \to 1} -(x-1) = -(1-1) = 0$$

# Section 2.2: The Derivative as a Function

If $f(x)$ is a function, then the set of numbers $f'(c)$ for various values of $c$ can define a new function. To proceed, we consider an alternative formulation for $f'(c)$.

If it exists, then $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$. Let $x = c+h$. Then $h = x-c$.

$\lim_{x\to c} h = \lim_{x\to c}(x-c) = c-c = 0$. So $x \to c$ is the same as $h \to 0$.

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{c+h-c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

# The Derivative Function

Let $f$ be a function. Define the new function $f'$ by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

called the **derivative** of $f$. The domain of this new function is the set

$$\{x | x \text{ is in the domain of } f, \text{ and } f'(x) \text{ exists}\}.$$

$f'$ is read as "$f$ prime."

# Example

Let $f(x) = \sqrt{x-1}$. Identify the domain of $f$. Find $f'$ and identify its domain.

For $x$ in the domain of $f$, we require $x-1 \geq 0 \Rightarrow x \geq 1$.

In interval notation, the domain is $[1, \infty)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}\right) \cdot \left(\frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}\right)$$

$$= \lim_{h\to 0} \frac{x+h-1-(x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{1}-\cancel{x}+\cancel{1}}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-1}+\sqrt{x-1}} = \frac{1}{\sqrt{x+0-1}+\sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x-1}+\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

$$\boxed{\text{so} \quad f'(x) = \frac{1}{2\sqrt{x-1}}.}$$

For $x$ in the domain of $f'$, we require $x-1>0$.

That is, $x>1$. In interval notation, the

domain is $(1,\infty)$.

# Example

Find the equation of the line tangent to the graph of $f(x) = \sqrt{x-1}$ at the point $(5, 2)$.

We need $m_{tan}$ for $c = 5$. We know $m_{tan} = f'(5)$

From the last example, $f'(x) = \dfrac{1}{2\sqrt{x-1}}$.

$$m_{tan} = f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$y - 2 = \frac{1}{4}(x-5) \implies y - 2 = \frac{1}{4}x - \frac{5}{4}$$

$$\implies y = \frac{1}{4}x - \frac{5}{4} + 2 = \frac{1}{4}x + \frac{3}{4}$$

$$\boxed{y = \frac{1}{4}x + \frac{3}{4}}$$

# Example

Is there any point on the graph of $f(x) = \sqrt{x-1}$ at which the tangent line is parallel to the line $8y = x$?

$8y = x \Rightarrow y = \frac{1}{8}x$. The question becomes "is there any point on the graph where $m_{tan} = \frac{1}{8}$?"

Well, $m_{tan} = f'(c)$ for $c$ in the domain of $f'$.

$f'(c) = \dfrac{1}{2\sqrt{c-1}}$. Set $\dfrac{1}{2\sqrt{c-1}} = \dfrac{1}{8}$.

Take reciprocals $\quad 8 = 2\sqrt{c-1}$

$$\Rightarrow \frac{8}{2} = \sqrt{c-1}$$

$$\Longrightarrow \quad 16 = c - 1 \quad \Longrightarrow \quad c = 17.$$

This is the x-value of such a point. The y-value is $f(17) = \sqrt{17 - 1} = \sqrt{16} = 4$

There is one such point, $(17, 4)$.

# Question

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = 2x^2$; determine $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$\vdots$$

$$= 4x$$

(a) $f'(x) = 4$

(b) $f'(x) = 2x$

(c) $f'(x) = \frac{2(x+h)^2 - 2x^2}{h}$

(d) $f'(x) = 4x$

# Question

$f'(x) = 4x$

Let $f(x) = 2x^2$. Find the equation of the line tangent to the graph of $f$ at the point $(2, f(2))$.

$f(2) = 2(2^2) = 2 \cdot 4 = 8$

$f'(2) = 4 \cdot 2 = 8$

(a) $y = 8x - 8$

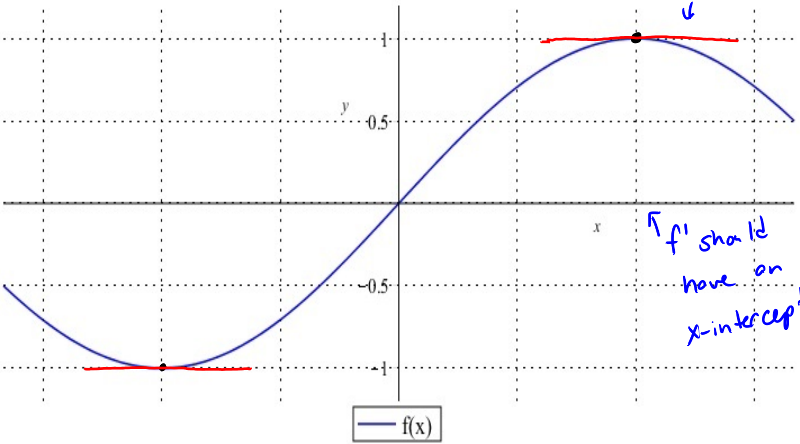
(b) $y = 4x(x-2) + 8$

(c) $y = 8x - 16$

(d) $y = 4x(x-2)$

# How are the functions $f(x)$ and $f'(x)$ related?

slope zero




f' should have an x-intercept here

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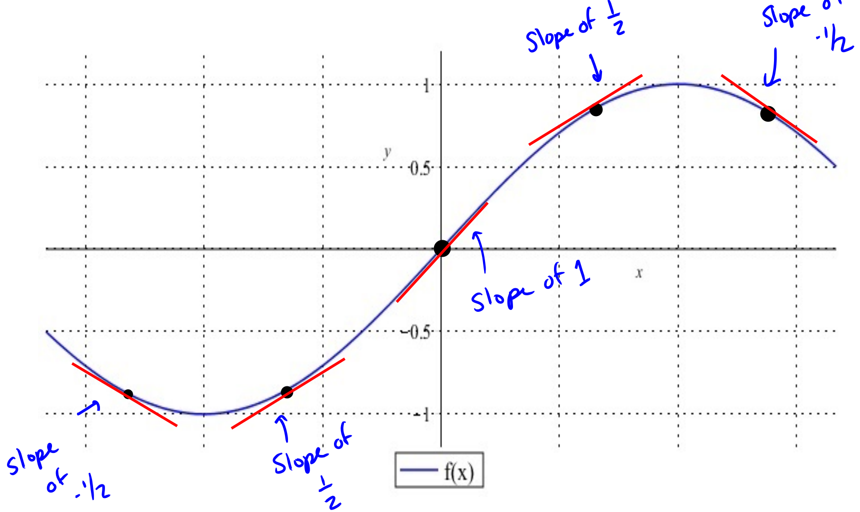

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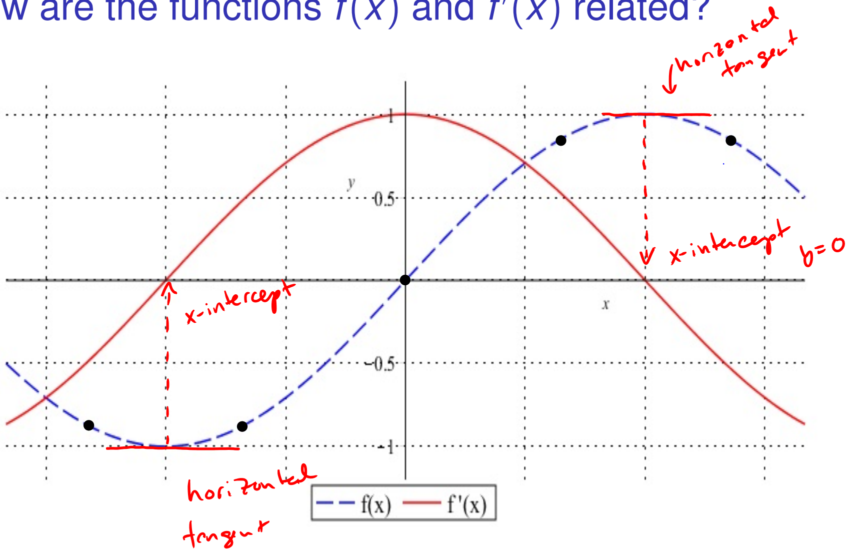

# Remarks:

- if $f(x)$ is a function of $x$, then $f'(x)$ is a new function of $x$ (called the derivative of $f$)
- The number $f'(c)$ (if it exists) is the slope of the curve of $y = f(x)$ at the point $(c, f(c))$
- this is also the slope of the tangent line to the curve of $y$ at $(c, f(c))$
- ”slope of the curve”, ”slope of the tangent line”, and ”rate of change” are the same concept

**Definition:** A function $f$ is said to be *differentiable* at $c$ if $f'(c)$ exists. It is called *differentiable* on an open interval $I$ if it is differentiable at each point in $I$.